

p -Adic Heisenberg Cantor sets, 2

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Abstract

In these informal notes, we continue to explore p -adic versions of Heisenberg groups and some of their variants, including the structure of the corresponding Cantor sets.

Contents

I	A broad view	2
1	Abelian groups	2
2	Non-abelian groups	4
3	Topological equivalence	6
4	Cauchy sequences	7
5	Cartesian products	8
6	An embedding	9
7	Coherent sequences	10
8	Completions	12
9	Haar measure	13
10	Heisenberg groups	15
11	Chains of subgroups	16
12	Haar measure, continued	18
13	Rings	20
14	A basic scenario	20

II Additional structure	22
15 Rings and modules	22
16 Heisenberg groups	23
17 Rings of fractions	25
18 Modules of fractions	27
19 Heisenberg groups of fractions	28
20 Restriction of scalars	28
21 A class of examples	29
22 Chains of submodules	30
23 Completeness	31
24 Chains of ideals	34
25 Ideals and submodules	35
26 Homomorphisms	36
27 Bilinear mappings	38
28 Stronger conditions	39
References	40

Part I

A broad view

1 Abelian groups

Let A be an abelian group, in which the group structure is written additively, and let $A_0 = A \supseteq A_1 \supseteq A_2 \supseteq \cdots$ be a decreasing chain of subgroups of A such that $\bigcap_{j=0}^{\infty} A_j = \{0\}$. As a basic example, one can take A to be the group \mathbf{Z} under addition, m to be a positive integer greater than or equal to 2, and $A_j = m^j \mathbf{Z}$ for each $j \geq 0$, the subgroup of \mathbf{Z} consisting of multiples of m^j .

There is a standard way in which to define a topology on A under these conditions, where a set $U \subseteq A$ is an open set if for each $x \in U$ we have that

$$(1.1) \quad x + A_j \subseteq U$$

for some $j \geq 0$. It is easy to see that this does define a topology on A , and that A is Hausdorff with respect to this topology, because of the condition that $\bigcap_{j=0}^{\infty} A_j = \{0\}$. One can also check that the group operations of addition and taking inverses are continuous with respect to this topology, so that A becomes a topological group.

If $x \in A$, then let $j(x)$ be the largest nonnegative integer such that $x \in A_{j(x)}$ when $x \neq 0$, and put $j(0) = +\infty$. Thus

$$(1.2) \quad j(-x) = j(x)$$

and

$$(1.3) \quad j(x+y) \geq \min(j(x), j(y))$$

for every $x, y \in A$. The second assertion reflects the fact that $x+y \in A_j$ when $x, y \in A_j$ for some $j \geq 0$.

Let $r_0 \geq r_1 \geq r_2 \geq \dots$ be a sequence of positive real numbers that converges to 0, and put

$$(1.4) \quad \rho(x) = r_{j(x)}$$

when $x \in A$ and $x \neq 0$, and $\rho(0) = 0$. Observe that

$$(1.5) \quad \rho(-x) = \rho(x)$$

and

$$(1.6) \quad \rho(x+y) \leq \max(\rho(x), \rho(y))$$

for every $x, y \in A$, by the corresponding properties of $j(x)$.

If $x, y \in A$, then put

$$(1.7) \quad d(x, y) = \rho(x - y).$$

This defines an ultrametric on A , which means that is a metric on A that satisfies

$$(1.8) \quad d(x, z) \leq \max(d(x, y), d(y, z))$$

for every $x, y, z \in A$. Of course, this is a stronger version of the usual triangle inequality. By construction, $d(x, y)$ is also invariant under translations on A , in the sense that

$$(1.9) \quad d(x - z, y - z) = d(x, y)$$

for every $x, y, z \in A$. It is easy to see that the topology on A determined by this ultrametric is the same as the one described earlier.

Observe that every open or closed ball in A centered at 0 with positive radius with respect to this ultrametric $d(x, y)$ is equal to A_j for some $j \geq 0$. This is a bit nicer when the r_j 's are strictly decreasing, in which case A_j can be expressed as a ball around 0 for each j . The situation is a bit better still when the A_j 's are strictly decreasing in j , which is to say that $A_j \neq A_{j+1}$ for each j .

Conversely, suppose that $d(x, y)$ is a translation-invariant ultrametric on A , and put

$$(1.10) \quad \rho(x) = d(x, 0)$$

for each $x \in A$. Thus $\rho(x)$ is a nonnegative real-valued function on A which is equal to 0 if and only if $x = 0$. Because of translation invariance, (1.7) holds for every $x, y \in A$, and one can get (1.5) and (1.6) from the symmetry of $d(x, y)$ and the ultrametric version of the triangle inequality for $d(x, y)$. Using the ultrametric version of the triangle inequality, we also get that open and closed balls in A centered at 0 and with positive radii with respect to $d(x, y)$ are subgroups of A . A sequence of these subgroups corresponding to a decreasing sequence of radii converging to 0 determines the same topology on A as $d(x, y)$ does, in the same way as before.

Note that open balls in any ultrametric space are both open and closed in the associated topology, and that closed balls are both open and closed as well. This is easy to check, using the ultrametric version of the triangle inequality. An open subgroup of a topological group is also closed, because its complement can be expressed as a union of translates of itself, and hence is an open set too. If A is equipped with the topology determined by the subgroups A_j as discussed at the beginning of the section, then each A_j is automatically an open subgroup of A , and thus closed as well.

2 Non-abelian groups

Let G be a group which is not necessarily abelian, with the group structure written multiplicatively, and with e as the identity element. As in the previous section, let $G_0 = G \supseteq G_1 \supseteq G_2 \supseteq \cdots$ be a decreasing chain of subgroups of G such that $\bigcap_{j=0}^{\infty} G_j = \{e\}$. Let us say that a set $U \subseteq G$ is an open set if for each $x \in U$ there is a $j \geq 0$ such that

$$(2.1) \quad xG_j \subseteq U.$$

It is easy to see that this defines a topology on G , and that this topology is Hausdorff, because $\bigcap_{j=0}^{\infty} G_j = \{e\}$.

If $x \in G$ and $x \neq e$, then let $j(x)$ be the largest nonnegative integer such that $x \in G_{j(x)}$, and put $j(e) = +\infty$. Observe that

$$(2.2) \quad j(x^{-1}) = j(x)$$

and that

$$(2.3) \quad j(xy) \geq \min(j(x), j(y))$$

for every $x, y \in G$, because the G_j 's are subgroups of G . Let $r_0 \geq r_1 \geq r_2 \geq \cdots$ be a sequence of positive real numbers that converges to 0, and put

$$(2.4) \quad \rho(x) = r_{j(x)}$$

for every $x \in G$ such that $x \neq e$, and $\rho(e) = 0$. Thus

$$(2.5) \quad \rho(x^{-1}) = \rho(x)$$

and

$$(2.6) \quad \rho(xy) \leq \max(\rho(x), \rho(y))$$

for every $x, y \in G$, by the corresponding properties of $j(x)$.

If $x, y \in G$, then put

$$(2.7) \quad d(x, y) = \rho(y^{-1}x).$$

It is easy to see that this defines an ultrametric on G , which defines the same topology on G as before. This ultrametric is also invariant under left translations on G , in the sense that

$$(2.8) \quad d(ax, ay) = d(x, y)$$

for every $a, x, y \in G$. In particular, $x \mapsto ax$ defines a homeomorphism on G for each $a \in G$. Of course, one can check directly that left translations determine homeomorphisms on G with respect to the topology described earlier.

Suppose for the moment that the G_j 's are normal subgroups of G , so that

$$(2.9) \quad j(axa^{-1}) = j(x)$$

for every $a, x \in G$. This implies that

$$(2.10) \quad \rho(axa^{-1}) = \rho(x)$$

for every $a, x \in G$, and hence that

$$(2.11) \quad d(xa, ya) = d(x, y)$$

for every $a, x, y \in G$. In particular, $x \mapsto xa$ is a homeomorphism on G for each $a \in G$ in this case, which could also be verified more directly.

Let us say that the chain of subgroups G_j is weakly normal in G if for each $a \in G$ and $j \geq 0$ there is an $l \geq 0$ such that

$$(2.12) \quad G_l \subseteq aG_ja^{-1}.$$

This implies that $x \mapsto xa$ is a homeomorphism on G for each $a \in G$, even if the ultrametric $d(x, y)$ may not be invariant under right translations. Alternatively, one can first check that $x \mapsto x^{-1}$ is continuous on G in this case, and then use this to derive the continuity of right translations from the continuity of left translations. One can also check that G is a topological group under these conditions.

Actually, the group operations on G are automatically continuous at the identity element, without this extra hypothesis. Note that G_j is an open set in G for each $j \geq 0$, by construction. Similarly, the left coset aG_j of G_j is an open set in G for each $a \in G$ and $j \geq 0$. As in the previous section, G_j is also a closed set in G for each j , because its complement is open.

Conversely, suppose that $d(x, y)$ is an ultrametric on G that is invariant under left translations, and put

$$(2.13) \quad \rho(x) = d(x, e)$$

for each $x \in G$. Thus $\rho(x)$ is a nonnegative real-valued function on G which is equal to 0 only when $x = 0$. Because $d(x, y)$ is invariant under left translations, (2.7) holds for every $x, y \in G$, with this definition of ρ . Similarly, (2.5) holds for every $x \in G$, because

$$(2.14) \quad \rho(x^{-1}) = d(x^{-1}, e) = d(e, x^{-1}) = d(x, e) = \rho(x).$$

This uses the symmetry of $d(\cdot, \cdot)$ in the second step, and invariance under left translations in the third. If $x, y \in G$, then

$$(2.15) \quad \rho(xy) = d(xy, e) \leq \max(d(xy, x), d(x, e))$$

by the ultrametric version of the triangle inequality. This implies that

$$(2.16) \quad \rho(xy) \leq \max(d(y, e), d(x, e)) = \max(\rho(y), \rho(x)),$$

using invariance under left translations again. This shows that (2.6) also holds for every $x, y \in G$. If $d(x, y)$ is invariant under right translations too, then it is easy to see that (2.10) holds for every $a, x \in G$ as well.

It follows from (2.5) and (2.6) that open and closed balls in G centered at e with positive radii with respect to $d(x, y)$ are subgroups of G . If $d(x, y)$ is also invariant under right translations on G , then (2.10) implies that these balls around e are normal subgroups of G . Using a decreasing sequence of radii that converges to 0, one gets a decreasing sequence of subgroups of G , as before. The topology on G associated to this sequence of subgroups is then the same as the topology determined by the given ultrametric $d(x, y)$.

3 Topological equivalence

Let G be a group, and let $G_0 = G \supseteq G_1 \supseteq G_2 \supseteq \cdots$ be a decreasing chain of subgroups of G such that $\bigcap_{j=1}^{\infty} G_j = \{e\}$, as in the previous section. Suppose that

$$(3.1) \quad \tilde{G}_0 = G \supseteq \tilde{G}_1 \supseteq \tilde{G}_2 \supseteq \cdots$$

is another decreasing chain of subgroups of G such that $\bigcap_{j=1}^{\infty} \tilde{G}_j = \{e\}$. Let us say that these two chains of subgroups of G are topologically equivalent if for each $j \geq 1$ there is an $l \geq 1$ such that

$$(3.2) \quad \tilde{G}_l \subseteq G_j,$$

and similarly for each $k \geq 1$ there is an $n \geq 1$ such that

$$(3.3) \quad G_n \subseteq \tilde{G}_k.$$

This is exactly the condition that ensures that the topologies on G associated to these two chains of subgroups as in the previous section are the same.

Suppose that the chain of subgroups of G given by the G_j 's is weakly normal, in the sense described in the preceding section. This implies that for each $a \in G$ and $j \geq 1$ there is a $k \geq 1$ such that

$$(3.4) \quad a G_k a^{-1} \subseteq G_j,$$

by applying the previous condition to a^{-1} . Of course,

$$(3.5) \quad a G_0 a^{-1} = G \supseteq a G_1 a^{-1} \supseteq a G_2 a^{-1} \supseteq \dots$$

is a decreasing chain of subgroups of G such that $\bigcap_{j=1}^{\infty} a G_j a^{-1} = \{e\}$ for each $a \in G$, because of the corresponding properties of the G_j 's. Thus weak normality may be reformulated as saying that (3.5) is topologically equivalent to the original chain of subgroups of G given by the G_j 's for each $a \in G$.

Suppose now that G_j has finite index in G for each j , and consider

$$(3.6) \quad H_j = \bigcap_{a \in G} a G_j a^{-1}.$$

This is automatically a normal subgroup of G , which is also a subgroup of G_j . Equivalently,

$$(3.7) \quad H_j = \bigcap_{a \in A_j} a G_j a^{-1},$$

where A_j is any collection of representatives of the left cosets of G_j in G , so that every element of G can be represented as $a x$ for some $a \in A_j$ and $x \in G_j$. If G_j has finite index in G , then we can take A_j to have only finitely many elements, and H_j is the intersection of only finitely many conjugates of G_j . If the chain of G_j 's is also weakly normal in G , then it follows that the chain of H_j 's is topologically equivalent to the chain of G_j 's in G .

4 Cauchy sequences

Let G be a group, and let $G_0 = G \supseteq G_1 \supseteq G_2 \supseteq \dots$ be a decreasing chain of subgroups of G such that $\bigcap_{j=0}^{\infty} G_j = \{e\}$, as in Section 2. Let us say that a sequence $\{x_j\}_{j=1}^{\infty}$ of elements of G is a Cauchy sequence if for each positive integer n there is an $L \geq 1$ such that

$$(4.1) \quad x_l^{-1} x_j \in G_n$$

for every $j, l \geq L$. This is equivalent to saying that $\{x_j\}_{j=1}^{\infty}$ is a Cauchy sequence with respect to any metric $d(x, y)$ on G that determines the same topology on G as in Section 2 and is invariant under left translations on G , including the ultrametrics discussed previously. If $\{x_j\}_{j=1}^{\infty}$ converges to an element x of G with respect to the topology defined in Section 2, then it is easy to see that $\{x_j\}_{j=1}^{\infty}$ is a Cauchy sequence in G . If every Cauchy sequence of elements of G converges to an element of G , then we say that G is complete.

Suppose that $\tilde{G}_0 = G \supseteq \tilde{G}_1 \supseteq \tilde{G}_2 \supseteq \cdots$ is another decreasing chain of subgroups of G such that $\bigcap_{j=1}^{\infty} \tilde{G}_j = \{e\}$. If the chain of \tilde{G}_j 's is topologically equivalent to the chain of G_j 's in G , then it follows that a sequence $\{x_j\}_{j=1}^{\infty}$ of elements of G is a Cauchy sequence with respect to the chain of G_j 's if and only if it is a Cauchy sequence with respect to the chain of \tilde{G}_j 's. In particular, G is complete with respect to the chain of G_j 's if and only if it is complete with respect to the chain of \tilde{G}_j 's.

Let us say that a sequence $\{x_j\}_{j=1}^{\infty}$ of elements of G is “strongly Cauchy” if

$$(4.2) \quad x_l \in x_j G_j$$

for every $l \geq j$. It is easy to see that strongly Cauchy sequences are Cauchy sequences, and that every Cauchy sequence has a subsequence that is strongly Cauchy. As usual, if a Cauchy sequence has a subsequence that converges, then the whole sequence converges to the same limit. It follows that G is complete if every strongly Cauchy sequence in G converges.

A pair of Cauchy sequences $\{x_j\}_{j=1}^{\infty}, \{y_j\}_{j=1}^{\infty}$ in G are said to be “equivalent” if $\{y_j^{-1} x_j\}_{j=1}^{\infty}$ converges to the identity element e of G . More explicitly, this means that for each $n \geq 1$ there is an $L \geq 1$ such that

$$(4.3) \quad y_j^{-1} x_j \in G_n$$

for every $j \geq L$. It is easy to see that this defines an equivalence relation on the set of Cauchy sequences in G , and that a Cauchy sequence converges if and only if it is equivalent to a constant sequence. Note that every subsequence of a Cauchy sequence $\{x_j\}_{j=1}^{\infty}$ is automatically a Cauchy sequence, which is equivalent to $\{x_j\}_{j=1}^{\infty}$. In particular, every Cauchy sequence in G is equivalent to a strongly Cauchy sequence, by the remarks in the previous paragraph.

5 Cartesian products

Let X_1, X_2, \dots be a sequence of nonempty sets, and let $X = \prod_{j=1}^{\infty} X_j$ be their Cartesian product. Thus X consists of the sequences $x = \{x_j\}_{j=1}^{\infty}$ such that $x_j \in X_j$ for each j . If $x, y \in X$ and $x \neq y$, then let $j(x, y)$ be the largest nonnegative integer such that $x_j = y_j$ when $j \leq j(x, y)$. Equivalently, $j(x, y) + 1$ is the smallest positive integer j such that $x_j \neq y_j$. In particular, $j(x, y) = 0$ when $x_1 \neq y_1$. If $x = y$, then put $j(x, y) = +\infty$. Observe that

$$(5.1) \quad j(x, y) = j(y, x)$$

and

$$(5.2) \quad j(x, z) \geq \min(j(x, y), j(y, z))$$

for every $x, y, z \in X$. More precisely, (5.2) reduces to the statement that $x_j = z_j$ for every $j \leq l$ when $x_j = y_j$ and $y_j = z_j$ for every $j \leq l$.

Let $r_0 \geq r_1 \geq r_2 \geq \cdots$ be a sequence of positive real numbers that converges to 0. If $x, y \in X$, then put

$$(5.3) \quad d(x, y) = r_{j(x, y)}$$

when $x \neq y$, and of course $d(x, y) = 0$ when $x = y$. Thus

$$(5.4) \quad d(x, z) \leq \max(d(x, y), d(y, z))$$

for every $x, y, z \in X$, because of (5.2). It follows that $d(x, y)$ is an ultrametric on X , and it is easy to see that the topology determined on X by $d(x, y)$ is the same as the product topology, using the discrete topology on X_j for each j . In particular, X is compact with respect to this topology when X_j has only finitely many elements for each j .

Suppose that $x(1) = \{x_j(1)\}_{j=1}^\infty, x(2) = \{x_j(2)\}_{j=1}^\infty, x(3) = \{x_j(3)\}_{j=1}^\infty, \dots$ is a sequence of elements of X which is a Cauchy sequence with respect to the ultrametric $d(x, y)$. It is easy to see that this implies that for each positive integer j there is an $x_j \in X_j$ such that $x_j(l) = x_j$ for all sufficiently large l . If $x = \{x_j\}_{j=1}^\infty$, then $x \in X$, and $x(1), x(2), x(3), \dots$ converges as a sequence of elements of X to x with respect to the ultrametric defined in the previous paragraph. Equivalently, $x(1), x(2), x(3), \dots$ converges to x with respect to the product topology on X , corresponding to the discrete topology on each X_j . This shows that X is complete as a metric space with respect to the ultrametric defined in the previous paragraph.

6 An embedding

Let G be a group, and let $G_0 = G \supseteq G_1 \supseteq G_2 \supseteq \dots$ be a decreasing chain of subgroups such that $\bigcap_{j=0}^\infty G_j = \{e\}$, as in Section 2. Even if the G_j 's are not normal subgroups of G , we can still consider the quotients G/G_j as sets, where G/G_j is the set of left cosets aG_j of G_j in G . Let q_j be the canonical quotient mapping from G onto G_j , which sends each $a \in G$ to the corresponding coset aG_j in G/G_j . Let X be the Cartesian product $\prod_{j=1}^\infty (G/G_j)$ of these quotient spaces, and let q be the mapping from G into X which sends $a \in G$ to the sequence $q(a) = \{q_j(a)\}_{j=1}^\infty$ of corresponding elements of the quotient spaces G/G_j . Of course, if G_j is a normal subgroup in G for each j , then the quotient G/G_j is a group in a natural way, and the quotient mapping q_j is a group homomorphism. In this case, we can also consider X as a group, where the group operations are defined coordinatewise. This is the same as the direct product of the groups G/G_j , $j \in \mathbf{Z}_+$, and the mapping q is then a homomorphism from G into X .

If $a, b \in G$ and $j \geq 1$, then $q_j(a) = q_j(b)$ if and only if $aG_j = bG_j$, which is equivalent to $b^{-1}a \in G_j$. Let $j_G(a)$ be defined for $a \in G$ as in Section 2, and let $j_X(x, y)$ be defined for $x, y \in X$ as in Section 5. Using the previous remark, it is easy to see that

$$(6.1) \quad j_G(b^{-1}a) = j_X(q(a), q(b))$$

for every $a, b \in G$. Let $r_0 \geq r_1 \geq r_2 \geq \dots$ be a sequence of positive real numbers that converges to 0, and let d_G and d_X be the corresponding ultrametrics on G and X defined in Sections 2 and 5, respectively. It follows from (6.1) that

$$(6.2) \quad d_G(a, b) = d_X(q(a), q(b))$$

for every $a, b \in G$. In particular, q defines a homeomorphism from G onto $q(G)$ in X , with respect to the topology on $q(G)$ induced by the one on X . Of course, one can also check this directly, using the original description of the topology on G associated to the G_j 's, and the product topology on X corresponding to the discrete topology on G/G_j for each j .

7 Coherent sequences

Let G be a group, and let $G_0 = G \supseteq G_1 \supseteq G_2 \supseteq \cdots$ be a decreasing chain of subgroups of G such that $\bigcap_{j=1}^{\infty} G_j = \{e\}$, as in Section 2. If $j \leq l$, then there is a canonical mapping $\theta_{j,l}$ from G/G_l onto G/G_j , which sends a coset xG_l in G/G_l to the corresponding coset xG_j in G/G_j . This is well-defined, because $G_l \subseteq G_j$ when $j \leq l$, by hypothesis. Note that

$$(7.1) \quad \theta_{j,l} \circ \theta_{l,n} = \theta_{j,n}$$

when $j \leq l \leq n$. If the G_k 's are all normal subgroups of G , so that G/G_k is a group for each k , then $\theta_{j,l}$ is a group homomorphism from G/G_l onto G/G_j when $j \leq l$.

Let $a = \{a_j\}_{j=1}^{\infty}$ be an element of the Cartesian product $\prod_{j=1}^{\infty} (G/G_j)$, so that $a_j \in G/G_j$ for each j . If

$$(7.2) \quad \theta_{j,l}(a_l) = a_j$$

for every $j \leq l$, then $a = \{a_j\}_{j=1}^{\infty}$ is said to be a coherent sequence. Of course, it suffices to check that (7.2) holds with $l = j + 1$ for each j .

Let q_j be the canonical quotient mapping from G onto G/G_j , as before. If $x \in G$, then $q(x) = \{q_j(x)\}_{j=1}^{\infty}$ is clearly a coherent sequence. If $\{x_j\}_{j=1}^{\infty}$ is a sequence of elements of G , then $\{q_j(x_j)\}_{j=1}^{\infty}$ is a coherent sequence if and only if $\{x_j\}_{j=1}^{\infty}$ is strongly Cauchy, in the sense of Section 4. Note that every element $\{a_j\}_{j=1}^{\infty}$ of $\prod_{j=1}^{\infty} (G/G_j)$ can be represented as $\{q_j(x_j)\}_{j=1}^{\infty}$ for some sequence $\{x_j\}_{j=1}^{\infty}$ of elements of G . Hence every coherent sequence $\{a_j\}_{j=1}^{\infty}$ can be represented as $\{q_j(x_j)\}_{j=1}^{\infty}$ for some strongly Cauchy sequence $\{x_j\}_{j=1}^{\infty}$ of elements of G .

If $\{x_j\}_{j=1}^{\infty}$ and $\{y_j\}_{j=1}^{\infty}$ are strongly Cauchy sequences of elements of G such that

$$(7.3) \quad q_j(x_j) = q_j(y_j)$$

for each j , then $\{x_j\}_{j=1}^{\infty}$ and $\{y_j\}_{j=1}^{\infty}$ are equivalent as Cauchy sequences in G , as in Section 4. Conversely, suppose that $\{x_j\}_{j=1}^{\infty}$ and $\{y_j\}_{j=1}^{\infty}$ are equivalent Cauchy sequences in G , so that $\{y_j^{-1}x_j\}_{j=1}^{\infty}$ converges to e in G . This implies that for each $j \geq 1$ there is an $L \geq 1$ such that

$$(7.4) \quad y_l^{-1}x_l \in G_j$$

for every $l \geq L$, as in (4.3). Equivalently,

$$(7.5) \quad x_l \in y_l G_j \quad \text{and} \quad y_l \in x_l G_j$$

for every $l \geq L$, so that

$$(7.6) \quad q_j(x_l) = q_j(y_l)$$

for every $l \geq L$. If $\{x_j\}_{j=1}^\infty$ and $\{y_j\}_{j=1}^\infty$ are strongly Cauchy sequences, then it follows that (7.3) holds for each j , because

$$(7.7) \quad q_j(x_l) = q_j(x_j) \quad \text{and} \quad q_j(y_l) = q_j(y_j)$$

when $l \geq j$.

If $\{x_j\}_{j=1}^\infty$ is an ordinary Cauchy sequence in G , then for each $n \geq 1$ there is an $L(n) \geq 1$ such that

$$(7.8) \quad x_l^{-1} x_j \in G_n$$

for every $j, l \geq L(n)$, and hence

$$(7.9) \quad x_j \in x_l G_n \quad \text{and} \quad x_l \in x_j G_j$$

for every $j, l \geq L(n)$. Equivalently,

$$(7.10) \quad q_n(x_j) = q_n(x_l)$$

for every $j, l \geq L(n)$. Put

$$(7.11) \quad a_n = q_n(x_{L(n)})$$

for each $n \geq 1$, and observe that $\{a_n\}_{n=1}^\infty$ is a coherent sequence. Let $\{y_j\}_{j=1}^\infty$ be another Cauchy sequence in G , and let $\{b_n\}_{n=1}^\infty$ be the corresponding coherent sequence. It is easy to see that $a_n = b_n$ for each n if and only if $\{x_j\}_{j=1}^\infty$ and $\{y_j\}_{j=1}^\infty$ are equivalent as Cauchy sequences in G , as in the previous paragraph.

Suppose that $\prod_{j=1}^\infty (G/G_j)$ is equipped with the product topology associated to the discrete topology on G/G_j for each j , as before. It is easy to see that the set of all coherent sequences is closed as a subset of $\prod_{j=1}^\infty (G/G_j)$ with this topology. Let us check that the image $q(G)$ of G under the usual embedding into $\prod_{j=1}^\infty (G/G_j)$ is dense in the set of coherent sequences. Let $a = \{a_j\}_{j=1}^\infty$ be a coherent sequence, let n be a positive integer, and let $x(n)$ be an element of G such that

$$(7.12) \quad q_n(x(n)) = a_n.$$

Under these conditions, we have that

$$(7.13) \quad q_j(x(n)) = a_j$$

for $j = 1, \dots, n$, and hence that $q(x(n))$ converges to a in $\prod_{j=1}^\infty (G/G_j)$ as $n \rightarrow \infty$, as desired.

If G is complete, then $q(G)$ is equal to the set of coherent sequences. To see this, let $a = \{a_j\}_{j=1}^\infty$ be a coherent sequence, and let $\{x_j\}_{j=1}^\infty$ be a sequence of elements of G such that $q_j(x_j) = a_j$ for each j . As mentioned earlier, $\{x_j\}_{j=1}^\infty$ is a strongly Cauchy sequence in G , which converges to an element x of G , because G is complete. It follows that $\{x_j\}_{j=1}^\infty$ is equivalent as a Cauchy sequence to the constant sequence whose terms are equal to x , so that $q(x) = a$, as desired.

Alternatively, one can use the completeness of G to show that $q(G)$ is a closed set in $\prod_{j=1}^{\infty} (G/G_j)$. This also uses the fact that q is an isometric embedding of G into $\prod_{j=1}^{\infty} (G/G_j)$ with respect to suitable metrics, as before. This implies that $q(G)$ is the same as the set of coherent sequences, by the remarks in the preceding paragraph.

Suppose that G_j is a normal subgroup of G for each j , so that G/G_j is a group for each j , and the quotient mapping q_j is a group homomorphism from G onto G/G_j for each j . As in the previous section, the product $\prod_{j=1}^{\infty} (G/G_j)$ is also a group, where the group operations are defined coordinatewise, and the usual embedding q of G into $\prod_{j=1}^{\infty} (G/G_j)$ is also a group homomorphism. In this case, $\prod_{j=1}^{\infty} (G/G_j)$ is a topological group with respect to the product topology associated to the discrete topology on each factor. Remember that q is a homeomorphism from G onto $q(G)$, with respect to the topology on $q(G)$ induced by the product topology on $\prod_{j=1}^{\infty} (G/G_j)$. The closure $\overline{q(G)}$ of $q(G)$ in $\prod_{j=1}^{\infty} (G/G_j)$ is a subgroup of $\prod_{j=1}^{\infty} (G/G_j)$ under these conditions, and a topological group with respect to the topology induced by the product topology on $\prod_{j=1}^{\infty} (G/G_j)$.

Of course, $\overline{q(G)}$ is the same as the set of all coherent sequences, as before. Similarly, the closure $\overline{q(G_l)}$ of $q(G_l)$ in $\prod_{j=1}^{\infty} (G/G_j)$ consists of the coherent sequences $a = \{a_j\}_{j=1}^{\infty}$ such that a_j corresponds to the identity element in G/G_j for each $j \leq l$. This is a normal subgroup of $\overline{q(G)}$ for each l , which is relatively open in $\overline{q(G)}$. Note that $\bigcap_{l=1}^{\infty} \overline{q(G_l)}$ consists of only the identity element in $\overline{q(G)}$. The topology on $\overline{q(G)}$ determined by the decreasing chain of subgroups $\overline{q(G_l)}$ as in Section 2 is the same as the topology induced on $\overline{q(G)}$ by the product topology on $\prod_{j=1}^{\infty} (G/G_j)$.

Using q , we get a natural homomorphism from G/G_l into $\overline{q(G)}/\overline{q(G_l)}$ for each l . This homomorphism is injective, because the intersection of $\overline{q(G_l)}$ with $q(G)$ is equal to $q(G_l)$ for each l . One can check that this homomorphism is surjective for each l as well, and hence an isomorphism. This is basically the same as the density of $q(G)$ in $\overline{q(G)}$ in this topology.

8 Completions

Let G be a group, and let $G_0 = G \supseteq G_1 \supseteq G_2 \supseteq \cdots$ be a decreasing chain of subgroups of G such that $\bigcap_{j=1}^{\infty} G_j = \{e\}$, as before. The completion \widehat{G} of G may be defined as the set of equivalence classes of Cauchy sequences of elements of G , using the notion of equivalence of Cauchy sequences described in Section 4. Of course, this reduces to G itself when G is complete, and otherwise there is a natural embedding of G into \widehat{G} , that sends every element g of G to the Cauchy sequence in which every term is equal to g . There is also a natural one-to-one correspondence between equivalence classes of Cauchy sequences in G and coherent sequences, as in the previous section, so that \widehat{G} can be identified with the set of coherent sequences. This is especially nice when G_j is a normal subgroup of G for each j , as discussed later in the previous section.

Even if the G_j 's are not normal subgroups of G , we can still consider G as a metric space with respect to an ultrametric $d_G(\cdot, \cdot)$ as in Section 2, and this completion of G is the same as the completion of G as a metric space. If X denotes the product $\prod_{j=1}^{\infty} (G/G_j)$, then we have seen that there is an analogous ultrametric $d_X(\cdot, \cdot)$ on X , such that the usual embedding q of G into X is an isometric embedding. We have also seen that X is complete as a metric space with respect to $d_X(\cdot, \cdot)$, so that the completion of G as a metric space can be identified with the closure $\overline{q(G)}$ of $q(G)$ in X . The topology on X determined by $d_X(\cdot, \cdot)$ is the product topology corresponding to the discrete topology on each factor, and the closure $\overline{q(G)}$ of $q(G)$ with respect to this topology is the set of all coherent sequences.

Let a be an element of G , and consider the corresponding left translation mapping $g \mapsto ag$ on G . It is easy to see that this mapping sends Cauchy sequences of elements of G to Cauchy sequences, and that it sends equivalent Cauchy sequences to equivalent Cauchy sequences. In particular, this follows from the fact that the ultrametric $d_G(\cdot, \cdot)$ mentioned in the preceding paragraph is invariant under left translations. At any rate, this implies that $g \mapsto ag$ has a natural extension to a mapping on the completion \hat{G} of G for each $a \in G$.

There is also an induced mapping on G/G_j for each j , which sends each left coset gG_j in G/G_j to agG_j . This leads to a mapping on $X = \prod_{j=1}^{\infty} (G/G_j)$, using the mappings on G/G_j on each coordinate. The restriction of this mapping on X to $q(G)$ corresponds exactly to the left translation $g \mapsto ag$ on G . This mapping on X is an isometry with respect to $d_X(\cdot, \cdot)$, and a homeomorphism with respect to the product topology associated to the discrete topology on each factor in particular. Of course, one can check directly that this mapping sends coherent sequences to coherent sequences.

9 Haar measure

Let G be a group, and let $G_0 = G \supseteq G_1 \supseteq G_2 \supseteq \cdots$ be a decreasing chain of subgroups of G such that $\bigcap_{j=1}^{\infty} G_j = \{e\}$, as usual. In this section, let us suppose also that G_j has finite index in G for each j . This implies that $X = \prod_{j=1}^{\infty} (G/G_j)$ is compact with respect to the product topology associated to the discrete topology on each factor. In particular, the closure $\overline{q(G)}$ of the image $q(G)$ of G under the standard embedding q of G in X is compact. If G is complete, then it follows that G is compact with respect to the topology determined by the G_j 's. Alternatively, the hypothesis that G_j have finite index in G for each j implies that G is totally bounded, with respect to any left-invariant metric that determines the same topology, as in Section 2. This is because G is the union of finitely many left translates of G_j for each j . This implies that the completion of G as a metric space is compact, and that G is compact when G is complete.

As in the previous section, there is a natural action of G on G/G_j for each j , corresponding to left translations on G . This leads to an action of G on X coordinatewise, which restricts to an action on $\overline{q(G)}$. There is a natural Borel

probability measure on $\overline{q(G)}$, which is invariant under this action of G on $\overline{q(G)}$. Let us begin with the corresponding invariant integral on $\overline{q(G)}$.

Let n be a positive integer, and let π_n be the obvious coordinate mapping from X onto G/G_n , which sends $a = \{a_j\}_{j=1}^\infty$ in X to $\pi_n(a) = a_n$. Also let A_n be a subset of $\overline{q(G)}$ such that the restriction of π_n to A_n is a one-to-one mapping onto G/G_n . One may as well take A_n to be a subset of $q(G)$, so that the elements of A_n correspond to representatives of the left cosets of G_n in G . Note that the number of elements of A_n is equal to the number $|G/G_n|$ of left cosets of G_n in G , which is the same as the index of G_n in G .

Let f be a continuous real or complex-valued function on $\overline{q(G)}$, and put

$$(9.1) \quad I(f, A_n) = \frac{1}{|G/G_n|} \sum_{a \in A_n} f(a),$$

which is the average of f over A_n . If A_1, A_2, A_3, \dots is a sequence of subsets of $\overline{q(G)}$ as in the previous paragraph, then we put

$$(9.2) \quad I(f) = \lim_{n \rightarrow \infty} I(f, A_n).$$

This is analogous to the definition of a Riemann integral as a limit of Riemann sums. More precisely, the continuity of f on $\overline{q(G)}$ and the compactness of $\overline{q(G)}$ implies that f is uniformly continuous with respect to any of the usual ultrametrics on X . Using this, one can check that the averages $I(f, A_n)$ form a Cauchy sequence in \mathbf{R} or \mathbf{C} , as appropriate, and that the limit (9.2) does not depend on the choice of the A_n 's.

One can also check that $I(f)$ is invariant under the action of G on $\overline{q(G)}$ described earlier. This does not quite work for $I(f, A_n)$, but instead the action of G has the effect of changing A_n to another set with analogous properties for each n . This implies that the limit is invariant under the action of G , because it does not depend on the choice of A_n 's, as in the preceding paragraph.

Of course, $I(f)$ defines a linear functional on the space of continuous real or complex-valued functions on $\overline{q(G)}$. This linear functional is clearly nonnegative, in the sense that $I(f) \geq 0$ when f is a nonnegative real-valued function on $\overline{q(G)}$. The Riesz representation theorem then leads to a nonnegative Borel measure μ on $\overline{q(G)}$ such that

$$(9.3) \quad I(f) = \int_G f d\mu$$

for every continuous function f on $\overline{q(G)}$. Note that $\mu(G) = 1$, because $I(f) = 1$ when f is the constant function equal to 1 on $\overline{q(G)}$. Thus μ is a probability measure on $\overline{q(G)}$, which is invariant under the natural action of G on $\overline{q(G)}$, because of the corresponding property of $I(f)$.

If f_l is any real or complex-valued function on G/G_l for some positive integer l , then $f_l \circ \pi_l$ defines a continuous function on $\overline{q(G)}$. In this case, it is easy to see that $I(f_l \circ \pi_l, A_l)$ reduces to the average of f_l on G/G_l , and in particular does not depend on the choice of the set A_l as before. Moreover, $I(f_l \circ \pi_l, A_n)$ also reduces to the average of f_l over G/G_l for any $n \geq l$ and any choice of

A_n as before. This is basically because every left coset of G_l in G is a union of pairwise-disjoint left cosets of G_n in G . More precisely, every left coset of G_l in G is the pairwise-disjoint union of $|G_j/G_n|$ left cosets of G_n in G when $n \geq l$, and $|G/G_n|$ is equal to the product of $|G/G_l|$ and $|G_l/G_n|$.

It follows that $I(f_l \circ \pi_l)$ also reduces to the average of f_l over G/G_l , by passing to the limit as $n \rightarrow \infty$. Left translations on G simply permute the left cosets of G_l in G , which does not affect the average of f_l over G/G_l . If f is any continuous real or complex-valued function on $\overline{q(G)}$, then f can be approximated uniformly on $\overline{q(G)}$ by functions of the form $f_l \circ \pi_l$, where f_l is a function on G/G_l , because of the uniform continuity of f mentioned earlier. Of course, $I(f)$ is a bounded linear functional on the space of continuous functions on $\overline{q(G)}$ with respect to the supremum norm, because

$$(9.4) \quad |I(f)| \leq \sup_{a \in \overline{q(G)}} |f(a)|$$

for every continuous function f on $\overline{q(G)}$. Thus $I(f)$ is uniquely determined by its restriction to the dense set of functions of the form $f_l \circ \pi_l$ on $\overline{q(G)}$.

10 Heisenberg groups

Let A and A' be abelian groups, and let $B(x, y)$ be an A' -valued function of $x, y \in A$ that is additive in each variable. More precisely, this means that

$$(10.1) \quad B(x + w, y) = B(x, y) + B(w, y)$$

and

$$(10.2) \quad B(x, y + z) = B(x, y) + B(x, z)$$

for every $w, x, y, z \in A$. Of course, the group operations on A and A' are expressed additively here. Note that

$$(10.3) \quad B(x, 0) + B(x, 0) = B(x, 0 + 0) = B(x, 0)$$

and hence $B(x, 0) = 0$ for every $x \in A$, and similarly $B(0, y) = 0$ for every $y \in A$.

Let $G = A \times A'$ as a set, and put

$$(10.4) \quad (x, s) \diamond (y, t) = (x + y, s + t + B(x, y))$$

for every $(x, s), (y, t) \in G$, so that \diamond defines a binary operation on G . If (w, r) is another element of G , then

$$\begin{aligned} (10.5) \quad & ((w, r) \diamond (x, s)) \diamond (y, t) \\ &= (w + x, r + s + B(w, x)) \diamond (y, t) \\ &= (w + x + y, r + s + t + B(w, x) + B(w + x, y)) \\ &= (w + x + y, r + s + t + B(w, x) + B(w, y) + B(x, y)), \end{aligned}$$

and similarly

$$\begin{aligned}
(10.6) \quad (w, r) \diamond ((x, s) \diamond (y, t)) \\
&= (w, r) \diamond (x + y, s + t + B(x, y)) \\
&= (w + x + y, r + s + t + B(w, x + y) + B(x, y)) \\
&= (w + x + y, r + s + t + B(w, x) + B(w, y) + B(x, y)),
\end{aligned}$$

so that \diamond is associative on G . Moreover,

$$(10.7) \quad (x, s) \diamond (0, 0) = (0, 0) \diamond (x, s) = (x, s)$$

for every $(x, s) \in G$, since $B(x, 0) = B(0, x) = 0$ for every $x \in A$, as before. Thus $(0, 0)$ is the identity element of G with respect to \diamond .

Observe that

$$(10.8) \quad (x, s) \diamond (-x, -s + B(x, x)) = (0, B(x, x) + B(x, -x)) = (0, 0),$$

and similarly

$$(10.9) \quad (-x, -s + B(x, x)) \diamond (x, s) = (0, B(x, x) + B(-x, x)) = (0, 0)$$

for every $(x, s) \in G$. Thus

$$(10.10) \quad (x, s)^{-1} = (-x, -s + B(x, x))$$

is the inverse element of G corresponding to (x, s) with respect to \diamond . This is a bit simpler when $B(x, x) = 0$ for every $x \in A$, in which case

$$(10.11) \quad (x, s)^{-1} = (-x, -s)$$

for every $(x, s) \in G$. At any rate, this shows that G is a group with respect to \diamond , which is not commutative when $B(x, y)$ is not symmetric in x and y .

11 Chains of subgroups

Let us continue with the same notations and hypotheses as in the previous section. Let $A_0 = A \supseteq A_1 \supseteq A_2 \supseteq \cdots$ be a decreasing chain of subgroups of A such that $\bigcap_{j=1}^{\infty} A_j = \{0\}$, and let $A'_0 = A' \supseteq A'_1 \supseteq A'_2 \supseteq \cdots$ be a decreasing chain of subgroups of A' such that $\bigcap_{j=1}^{\infty} A'_j = \{0\}$. Suppose that

$$(11.1) \quad B(x, y) \in A'_j \quad \text{for every } x, y \in A_j,$$

in which case it is easy to see that $G_j = A_j \times A'_j$ is a subgroup of $G = A \times A'$ with respect to \diamond for each j . This puts us into the same situation as before, with a decreasing chain $G_0 = G \supseteq G_1 \supseteq G_2 \supseteq \cdots$ of subgroups of G whose intersection $\bigcap_{j=1}^{\infty} G_j$ consists of only the identity element $(0, 0)$ in G .

Observe that

$$\begin{aligned}
(11.2) \quad & ((x, s) \diamond (y, t)) \diamond (x, s)^{-1} \\
&= (x + y, s + t + B(x, y)) \diamond (-x, -s + B(x, x)) \\
&= (y, t + B(x, y) + B(x + y, -x) + B(x, x)) \\
&= (y, t + B(x, y) - B(y, x))
\end{aligned}$$

for every $(x, s), (y, t) \in G$. If

$$(11.3) \quad B(x, y) - B(y, x) \in A'_j$$

for every $x \in A$ and $y \in A_j$, then it follows that G_j is a normal subgroup of G for each j . In particular, this holds when $B(x, y)$ and $B(y, x)$ are both elements of A_j for every $x \in A$ and $y \in A_j$.

Suppose instead that for each $j \geq 1$ there is an $l(j) \geq j$ such that (11.3) holds for every $x \in A$ and $y \in A_{l(j)}$. In this case,

$$(11.4) \quad \tilde{G}_j = A_{l(j)} \times A'_j$$

is a normal subgroup of G for each j , and

$$(11.5) \quad G_{l(j)} \subseteq \tilde{G}_j \subseteq G_j$$

for each j . We can extend this to $j = 0$ by putting $l(0) = 0$, so that

$$(11.6) \quad \tilde{G}_0 = G \supseteq \tilde{G}_1 \supseteq \tilde{G}_2 \supseteq \cdots$$

is a decreasing chain of normal subgroups of G that satisfies $\bigcap_{j=1}^{\infty} \tilde{G}_j = \{(0, 0)\}$. Under these conditions, the chain of G_j 's is topologically equivalent to the chain of \tilde{G}_j 's, as in Section 3.

As a variant of this type of condition, suppose that for each $j \geq 1$ there is an $l(j) \geq j$ such that

$$(11.7) \quad B(w, z) \in A'_j$$

for every $w \in A$ and $z \in A_{l(j)}$. Observe that

$$\begin{aligned}
(11.8) \quad & (y, t)^{-1} \diamond (x, s) = (-y, -t + B(y, y)) \diamond (x, s) \\
&= (x - y, s - t + B(y, y) - B(x, y)) \\
&= (x - y, s - t + B(y, y - x))
\end{aligned}$$

for every $(x, s), (y, t) \in G$. If $x - y \in A_{l(j)}$ and $s - t \in A'_j$, then it follows that (11.8) is an element of $A_{l(j)} \times A'_j$. Conversely, if (11.8) is an element of $A_{l(j)} \times A'_j$, then $x - y \in A_{l(j)}$, which implies that

$$(11.9) \quad B(y, y - x) \in A'_j,$$

as in (11.7). Thus $s - t \in A'_j$, since $s - t - B(y, y - x) \in A'_j$, by hypothesis.

Of course, we can consider $A \times A'$ as an abelian group, with respect to coordinatewise addition, and we can consider

$$(11.10) \quad A_0 \times A'_0 = A \times A' \supseteq A_1 \times A'_1 \supseteq A_2 \times A'_2 \supseteq \cdots$$

as a decreasing chain of subgroups of $A \times A'$ such that $\bigcap_{j=1}^{\infty} A_j \times A'_j = \{(0, 0)\}$, as usual. The remarks in the previous paragraph show that the topology on $A \times A'$ as an abelian group with respect to this chain of subgroups as in Section 1 is the same as the topology on $G = A \times A'$ with \diamond as the group operation and with respect to the chain of subgroups $G_j = A_j \times A'_j$, under the condition (11.7). More precisely, if $A \times A'$ is equipped with a translation-invariant metric as in Section 1, and if G is equipped with a metric invariant under left translations with respect to \diamond as in Section 2, then the remarks in the previous paragraph imply that the identity mapping is uniformly continuous as a mapping from the former to the latter, and from the latter to the former, under these conditions.

In particular, a sequence $\{(x_k, s_k)\}_{k=1}^{\infty}$ of elements of $A \times A'$ is a Cauchy sequence in $A \times A'$ as an abelian group with the chain of subgroups $A_j \times A'_j$ if and only if it is a Cauchy sequence in $G = A \times A'$ with \diamond as the group operation and with respect to the chain of subgroups $G_j = A_j \times A'_j$ in this case. Of course, $\{(x_k, s_k)\}_{k=1}^{\infty}$ is a Cauchy sequence in $A \times A'$ as an abelian group if and only if $\{x_k\}_{k=1}^{\infty}$ and $\{s_k\}_{k=1}^{\infty}$ are Cauchy sequences in A and A' , respectively, as abelian groups with the chains of subgroups given by the A_j 's and the A'_j 's. If A and A' are complete as abelian groups with respect to these chains of subgroups, then it follows that $A \times A'$ is complete as an abelian group with respect to the chain of subgroups $A_j \times A'_j$, and that $G = A \times A'$ is complete with \diamond as the group operation \diamond with respect to the chain of subgroups $G_j = A_j \times A'_j$.

It is natural to combine these regularity conditions, and ask that for each $j \geq 1$ there be an $l(j) \geq 1$ such that

$$(11.11) \quad B(w, z), B(z, w) \in A'_j$$

for every $w \in A$ and $z \in A_{l(j)}$. This implies that (11.3) holds for every $x \in A$ and $y \in A_{l(j)}$, as well as including the previous condition that (11.7) hold for every $w \in A$ and $z \in A_{l(j)}$. Using this condition, one can also check that if $\{x_k\}_{k=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty}$ are Cauchy sequences in A as an abelian group with respect to the chain of subgroups A_j , then $\{B(x_k, y_k)\}_{k=1}^{\infty}$ is a Cauchy sequence in A' as an abelian group with respect to the chain of subgroups A'_j . Similarly, if $\{w_k\}_{k=1}^{\infty}$ and $\{z_k\}_{k=1}^{\infty}$ are Cauchy sequences in A that are equivalent to $\{x_k\}_{k=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty}$, respectively, then it is easy to see that $\{B(w_k, z_k)\}_{k=1}^{\infty}$ is equivalent to $\{B(x_k, y_k)\}_{k=1}^{\infty}$ as a Cauchy sequence in A' . This permits one to extend B to the completion of A with values in the completion of A' , from which one can then get the completion of G .

12 Haar measure, continued

Let us continue to use the same notation and hypotheses as in Section 10. If \tilde{A} and \tilde{A}' are subgroups of finite index in A and A' , respectively, then $\tilde{A} \times \tilde{A}'$ is a

subgroup of finite index in $A \times A'$, where $A \times A'$ is an abelian group with the group operations defined coordinatewise. More precisely, one can identify the quotient $(A \times A')/(\tilde{A} \times \tilde{A}')$ with the product of the quotients A/\tilde{A} and A'/\tilde{A}' , so that the index of $\tilde{A} \times \tilde{A}'$ in $A \times A'$ is equal to the product of the indices of \tilde{A} in A and \tilde{A}' in A' .

Suppose that $B(x, y) \in \tilde{A}'$ for every $x, y \in \tilde{A}$, so that $\tilde{G} = \tilde{A} \times \tilde{A}'$ is a subgroup of $G = A \times A'$ with respect to \diamond . Note that $\tilde{G}_1 = \tilde{A} \times A'$ is a normal subgroup of G , and that G/\tilde{G}_1 is isomorphic to A/\tilde{A} . Similarly, \tilde{G} is a normal subgroup of \tilde{G}_1 , and \tilde{G}_1/\tilde{G} is isomorphic to A'/\tilde{A}' . Thus \tilde{G} has finite index in G , equal to the product of the indices of \tilde{A} and \tilde{A}' in A and A' , respectively.

Let $A_0 = A \supseteq A_1 \supseteq A_2 \supseteq \dots$ and $A'_0 = A' \supseteq A'_1 \supseteq A'_2 \supseteq \dots$ be decreasing chains of subgroups of A and A' , respectively, such that $\bigcap_{j=1}^{\infty} A_j = \{0\}$ and $\bigcap_{j=1}^{\infty} A'_j = \{0\}$, as in the previous section. Suppose that A_j and A'_j have finite index in A and A' for each j , respectively, and that A and A' are complete with respect to these chains of subgroups. Thus A and A' are compact abelian topological groups with respect to the topologies determined by these chains of subgroups, and they have nice translation-invariant Borel probability measures, as in Section 9.

Suppose also that (11.1) holds, so that $G_j = A_j \times A'_j$ defines a decreasing chain of subgroups of G with respect to \diamond that satisfies $\bigcap_{j=1}^{\infty} G_j = \{(0, 0)\}$, as before. Note that G_j has finite index in G for each j , because of the corresponding properties of the A_j 's and A'_j 's, by the remarks at the beginning of the section. In addition, let us ask that for each $j \geq 1$ there be an $l(j) \geq 1$ such that (11.11) holds for every $w \in A$ and $z \in A_{l(j)}$. Under these conditions, G is a topological group with respect to the topology determined by the G_j 's as in Section 2, and this topology is the same as the one on $A \times A'$ as an abelian group corresponding to the subgroups $A_j \times A'_j$. Of course, the latter is the same as the product topology on $A \times A'$ corresponding to the topologies on A and A' as abelian groups determined by the subgroups A_j and A'_j , respectively.

In particular, G is compact with respect to the topology determined by the G_j 's in this case, because A and A' are compact. Using the translation-invariant Borel probability measures on A and A' mentioned earlier, one gets a Borel probability measure on the product $A \times A'$ that is invariant under translations on $A \times A'$ as an abelian group, with addition defined coordinatewise. A standard argument shows that this measure is also invariant under translations on $G = A \times A'$ with respect to \diamond , on the left and on the right. This is more easily seen in terms of integration of continuous functions on G , by integrating first over A' , and then over A .

13 Rings

Let R be a ring, and let $\mathcal{I}_0 = R \supseteq \mathcal{I}_1 \supseteq \mathcal{I}_2 \supseteq \cdots$ be a decreasing chain of ideals in R such that

$$(13.1) \quad \bigcap_{j=1}^{\infty} \mathcal{I}_j = \{0\}.$$

Although we are especially interested in commutative rings here, one can also consider non-commutative rings, in which case one should specify whether the ideals are one-sided or two-sided. At any rate, R is in particular an abelian group with respect to addition, and the \mathcal{I}_j 's form a decreasing chain of subgroups of R , as in Section 1. This leads to a topology on R , as before, and it is easy to see that multiplication on R is continuous with respect to this topology when R is commutative, or at least when the \mathcal{I}_j 's are two-sided ideals in R , so that R becomes a topological ring. If the \mathcal{I}_j 's are all left ideals, then $x \mapsto ax$ is a continuous mapping on R for every $a \in R$, and similarly for the case where the \mathcal{I}_j 's are all right ideals. The completion \widehat{R} of R can be defined as before, and multiplication on R can be extended to \widehat{R} to get a ring when the \mathcal{I}_j 's are two-sided ideals in R . If the \mathcal{I}_j 's are left ideals in R , then $x \mapsto ax$ can be extended to \widehat{R} for each $a \in R$, and similarly for the case where the \mathcal{I}_j 's are right ideals. If R is a commutative ring, then \widehat{R} is a commutative ring as well.

14 A basic scenario

Let R be a commutative ring, and let N be a positive integer. Thus $A = R^N$ is a commutative group, where addition is defined coordinatewise. We can also think of A as a module over R , with respect to coordinatewise multiplication by elements of R . Let $b = \{b_{p,q}\}_{p,q=1}^N$ be an $N \times N$ matrix with entries in R , and put

$$(14.1) \quad B(x, y) = \sum_{p=1}^N \sum_{q=1}^N b_{p,q} x_p y_q$$

for each $x = (x_1, \dots, x_N), y = (y_1, \dots, y_N) \in R^N$. This takes values in $A' = R$, and satisfies the additivity conditions (10.1) and (10.2) in each variable. This is also R -linear in each variable, in the sense that

$$(14.2) \quad B(rx, y) = B(x, ry) = r B(x, y)$$

for every $x, y \in R^N$ and $r \in R$. Conversely, if R has a multiplicative identity element, then any mapping from $R^N \times R^N$ into R with these properties is of the form (14.1).

This leads to a group structure \diamond on $G = A \times A' = R^N \times R$, as in Section 10. If $r \in R$, then put

$$(14.3) \quad \delta_r((x, s)) = (rx, r^2 s)$$

for each $(x, s) \in G$. It is easy to see that δ_r defines a homomorphism from G into itself for each $r \in R$, and that

$$(14.4) \quad \delta_r \circ \delta_{\tilde{r}} = \delta_{r\tilde{r}}$$

for every $r, r' \in R$.

Let $\mathcal{I}_0 = R \supseteq \mathcal{I}_1 \supseteq \mathcal{I}_2 \supseteq \cdots$ be a decreasing chain of ideals in R such that $\bigcap_{j=1}^{\infty} \mathcal{I}_j = \{0\}$, as in the previous section. We also ask that

$$(14.5) \quad xy \in \mathcal{I}_{j+l}$$

for every $x \in \mathcal{I}_j$ and $y \in \mathcal{I}_l$, and for any $j, l \geq 0$. Note that this condition holds automatically when \mathcal{I}_j is the “ j th power” of \mathcal{I}_1 , which is the ideal consisting of all finite sums of products of j elements of \mathcal{I}_1 . If R is the ring \mathbf{Z} of integers, for instance, and \mathcal{I}_1 is the ideal consisting of integers divisible by some integer $m \geq 2$, then the j th power of \mathcal{I}_1 is the ideal consisting of integers divisible by m^j for each $j \geq 2$. In this case, the completion \hat{R} of R is the usual m -adic completion of \mathbf{Z} , also known as the p -adic completion of \mathbf{Z} when m is a prime number p .

Of course, $A_j = \mathcal{I}_j^N$ is a subgroup of $A = R^N$ as an abelian group with respect to coordinatewise addition for each j . The A_j ’s form a decreasing chain of subgroups of A , for which the corresponding topology is the same as the product topology associated to the analogous topology on R for each factor.

Under these conditions, we have that

$$(14.6) \quad B(x, y) \in \mathcal{I}_{j+l}$$

for every $x \in \mathcal{I}_j$ and $y \in \mathcal{I}_l$, and for any $j, l \geq 0$. This implies that

$$(14.7) \quad G_j = A_j \times \mathcal{I}_{2j} = \mathcal{I}_j^N \times \mathcal{I}_{2j}$$

is a subgroup of $G = R^N \times R$ with respect to \diamond for each $j \geq 0$. As usual, the G_j ’s form a decreasing chain of subgroups of G , whose intersection consists of only the identity element $(0, 0)$ in G .

Alternatively,

$$(14.8) \quad H_j = A_j \times \mathcal{I}_j = \mathcal{I}_j^N \times \mathcal{I}_j$$

is a normal subgroup of G with respect to \diamond for each $j \geq 0$. The H_j ’s also form a decreasing chain of subgroups of G , with $\bigcap_{j=1}^{\infty} H_j = \{(0, 0)\}$. Note that

$$(14.9) \quad H_{2j} \subseteq G_j \subseteq H_j$$

for each j , so that the chains of G_j ’s and H_j ’s are topologically equivalent in G , as in Section 3.

To be more consistent with the notation used earlier, put $A'_j = \mathcal{I}_{2j}$ for each j , so that $G_j = A_j \times A'_j$. In this case, the regularity conditions discussed in Section 11 hold with $l(j) = 2j$. In particular, $\tilde{G}_j = A_{2j} \times A'_j$ is a normal subgroup of G with respect to \diamond for each j . Of course, this is the same as H_{2j} in the present notation.

The fact that H_j is a normal subgroup of G with respect to \diamond for each j only uses the hypothesis that \mathcal{I}_j be an ideal in R for each j , and not (14.5). Similarly, (14.5) is not needed for the compatibility conditions discussed in Section 11, to get in particular that the topology on G determined by the H_j 's is the same as the product topology on $G = R^N \times R$ associated to the topology on R determined by the \mathcal{I}_j 's. Using (14.5), we get the smaller subgroups G_j , which are equivalent topologically, but correspond to a different type of geometry. We also get that the G_j 's behave well with respect to dilations, in the sense that

$$(14.10) \quad \delta_r(G_j) \subseteq G_{j+l}$$

for every $r \in \mathcal{I}_l$ and any $l \geq 0$.

Part II

Additional structure

15 Rings and modules

Let R be a commutative ring, and let M be a module over R . Thus M is an abelian group with the group operation expressed additively, and for each $r \in R$ and $x \in M$ there is an element $rx \in M$ which satisfies the usual properties with respect to addition and multiplication on R and addition on M . These properties may be summarized by saying that we have a homomorphism from R into the ring of endomorphisms on M as an abelian group. If there is a multiplicative identity element e in R , then it is customary to ask that $ex = x$ for every $x \in M$, so that multiplication by e corresponds to the identity mapping on M .

Of course, R may be considered as a module over itself, using multiplication on R as a ring to define multiplication by elements of R on R as a module. Similarly, ideals in R may be considered as modules over R , and the quotient of R by an ideal is also a module over R in a natural way. Any abelian group A may be considered as a module over the ring \mathbf{Z} of integers, where na is the sum of n a 's when n is a positive integer, $0a = 0$, and $(-1)a = -a$ for every $a \in A$. If R is a field, then a module over R is the same as a vector space over R .

Suppose that M and N are modules over the same commutative ring R . A mapping f from M into N is said to be a module homomorphism or R -linear if f is a homomorphism from M into N as abelian groups, so that

$$(15.1) \quad f(x + y) = f(x) + f(y)$$

for every $x, y \in M$, and if

$$(15.2) \quad f(rx) = rf(x)$$

for every $r \in R$ and $x \in M$. If R is a field, then this is the same as a linear mapping between vector spaces. If f and g are homomorphisms from M into

N as R -modules and $r \in R$, then it is easy to see that $f + g$ and rf are also homomorphisms from M into N , where $f + g$ and rf are defined pointwise in the usual way. The set $\text{Hom}(M, N)$ of homomorphisms from M into N is thus a module over R , which may also be denoted $\text{Hom}_R(M, N)$, to make the choice of ring R explicit.

Suppose now that M_1 , M_2 , and N are modules over R . A mapping B from $M_1 \times M_2$ into N is said to be R -bilinear if it is R -linear in each coordinate separately. More precisely, this means that $B(x, y)$ is R -linear as a function of x from M_1 into N for each $y \in M_2$, and that $B(x, y)$ is R -linear as a function of y from M_2 into N for each $x \in M_1$. If R is a field, then this is the same as the usual notion of bilinear mappings for vector spaces. We shall be especially interested here in the case where $M_1 = M_2$.

As a basic class of examples, let n be a positive integer, and let $M_1 = M_2$ be the free module R^n of n -tuples of elements of R , where addition and multiplication by elements of R are defined coordinatewise on R^n . Let $b = \{b_{j,l}\}_{j,l=1}^n$ be an $n \times n$ matrix with entries in R , and put

$$(15.3) \quad B(x, y) = \sum_{j=1}^n \sum_{l=1}^n b_{j,l} x_j y_l$$

for every $x, y \in R^n$. It is easy to see that this is R -bilinear as a mapping from $R^n \times R^n$ into R . Conversely, if R has a multiplicative identity element, then every R -bilinear mapping from $R^n \times R^n$ into R is of this form.

Let M be a module over a commutative ring R . By a submodule of M we mean a subgroup M' of M as an abelian group such that $rx \in M'$ for every $x \in M'$ and $r \in R$. If we think of R as a module over itself, then submodules of R are the same as ideals. If R is a field, so that a module M over R is the same as a vector space over R , then a submodule of M is the same as a linear subspace of M . If R is any commutative ring, M and N are modules over R , and if f is a module homomorphism from M into N , then the kernel of f is a submodule of M , and f maps M onto a submodule of N .

Conversely, let M be a module over a commutative ring R , and let M' be a submodule of M . Because M' is a subgroup of M as an abelian group, we can first form the quotient M/M' as an abelian group. It is easy to see that multiplication by elements of R is well-defined on M/M' , so that M/M' is also a module over R , and the natural quotient mapping from M onto M/M' is a module homomorphism. Of course, the kernel of this quotient mapping is equal to M' by construction, so that every submodule is the kernel of a module homomorphism.

16 Heisenberg groups

Let R be a commutative ring, let M and N be R -modules, and let B be an R -bilinear mapping from $M \times M$ into R . Consider the binary operation \diamond on

$G = M \times N$ defined by

$$(16.1) \quad (x, s) \diamond (y, t) = (x + y, s + t + B(x, y)).$$

One can check that this defines a group structure on G , with $(0, 0)$ as the identity element, and

$$(16.2) \quad (x, s)^{-1} = (-x, -s + B(x, x))$$

as the inverse of (x, s) . If $r \in R$, then

$$(16.3) \quad \delta_r((x, s)) = (rx, r^2s)$$

defines a mapping from G into itself, which is a group homomorphism with respect to \diamond . Observe also that

$$(16.4) \quad \delta_{r_1} \circ \delta_{r_2} = \delta_{r_1 r_2}$$

for every $r_1, r_2 \in R$. As in the previous section, if R has a multiplicative identity element e , then it is customary to ask that multiplication by e act by the identity mapping on M and N . In this case, δ_e is also the identity mapping on G .

Suppose that M' and N' are submodules of M and N , respectively, such that

$$(16.5) \quad B(x, y) \in N'$$

for every $x, y \in M'$. Under these conditions, $M' \times N'$ is a subgroup of G with respect to \diamond , and

$$(16.6) \quad \delta_r(M' \times N') \subseteq M' \times N'$$

for every $r \in R$. In order to get a normal subgroup of G with respect to \diamond , we should also ask that

$$(16.7) \quad B(x, y) - B(y, x) \in N'$$

for every $x \in M$ and $y \in M'$. This is because

$$(16.8) \quad \begin{aligned} & ((x, s) \diamond (y, t)) \diamond (x, s)^{-1} \\ &= (x + y, s + t + B(x, y)) \diamond (-x, -s + B(x, x)) \\ &= (y, t + B(x, y) + B(x, x) + B(x + y, -x)) \\ &= (y, t + B(x, y) - B(y, x)) \end{aligned}$$

for every $(x, s), (y, t) \in G$.

Suppose instead that

$$(16.9) \quad B(x, y) \text{ and } B(y, x) \in N',$$

for every $x \in M$ and $y \in M'$, which implies (16.7). If $\widetilde{M} = M/M'$ and $\widetilde{N} = N/N'$ are the corresponding quotient modules, then it is easy to see that there is an R -bilinear mapping \widetilde{B} from $\widetilde{M} \times \widetilde{M}$ into \widetilde{N} that corresponds to B . This leads to a group structure $\widetilde{\diamond}$ on $\widetilde{G} = \widetilde{M} \times \widetilde{N}$, as before. Let q_M and q_N be the usual quotient mappings from M and N onto \widetilde{M} and \widetilde{N} , respectively, and let q

be the mapping which sends (x, s) in G to $(q_M(x), q_N(s))$ in \tilde{G} . By construction, q is a homomorphism from G onto \tilde{G} with respect to \diamond and $\tilde{\diamond}$, respectively, and the kernel of q is equal to $M' \times N'$. Because \tilde{M} and \tilde{N} are R -modules, we can define $\tilde{\delta}_r$ on \tilde{G} in the same way that δ_r was defined on G , and with analogous properties. This homomorphism $q : G \rightarrow \tilde{G}$ also intertwines δ_r and $\tilde{\delta}_r$ for each $r \in R$, in the sense that

$$(16.10) \quad q \circ \delta_r = \tilde{\delta}_r \circ q.$$

Alternatively, suppose that M_1 and N_1 are R -modules, and that $\phi : M \rightarrow M_1$ and $\psi : N \rightarrow N_1$ are module homomorphisms. Suppose also that B_1 is an R -bilinear mapping from $M_1 \times N_1$ into N_1 , and that

$$(16.11) \quad B_1(\phi(x), \phi(y)) = \psi(B(x, y))$$

for every $x, y \in M$. This leads to a group structure \diamond_1 on $G_1 = M_1 \times N_1$, as before, and a semigroup of compatible dilations on G_1 associated to elements of R as in (16.3). It is easy to see that the mapping from (x, s) in G to $(\phi(x), \psi(s))$ in G_1 is a group homomorphism that intertwines the corresponding families of dilations. If M' and N' are the kernels of ϕ and ψ , then (16.11) implies that (16.9) holds for every $x \in M$ and $y \in M'$.

17 Rings of fractions

Let R be a commutative ring with multiplicative identity element e . Note that e is allowed to be equal to 0, in which case $R = \{0\}$. If R is an integral domain, which is to say that $R \neq \{0\}$ and there are no nontrivial zero divisors in R , then the corresponding field of fractions can be defined in a standard way. There are also versions of this that include the possibility of nontrivial zero divisors, and which are more precise about the elements of the ring that are allowed as denominators in the fractions. We shall review these matters in this section, following the treatment in Chapter 3 of [1].

Let S be a multiplicatively closed subset of R , in the sense that $e \in S$ and $st \in S$ for every $s, t \in S$. Equivalently, S is a sub-semi-group of R as a semigroup with respect to multiplication, which includes the multiplicative identity element e . Consider the relation \equiv defined on $R \times S$ by saying that $(a, s) \equiv (b, t)$ when

$$(17.1) \quad (at - bs)v = 0$$

in R for some $v \in S$. It is easy to see that this relation is reflexive and symmetric on $R \times S$, and we would like to check that it is transitive. If (a, s) , (b, t) , and (c, u) are elements of $R \times S$ satisfy $(a, s) \equiv (b, t)$ and $(b, t) \equiv (c, u)$, then there are $v, w \in S$ such that (17.1) and

$$(17.2) \quad (bu - ct)w = 0$$

hold. Multiplying (17.1) by uw and (17.2) by sv , and then adding the resulting equations, we get that

$$(17.3) \quad atvuw - ctws v = 0.$$

This is the same as

$$(17.4) \quad (a u - c s) t v w = 0,$$

which implies that $(a, s) \equiv (c, u)$, as desired, since $t v w \in S$.

Thus \equiv defines an equivalence relation on $R \times S$, and we let $S^{-1} R$ denote the collection of corresponding equivalence classes in $R \times S$. If $(a, s) \in R \times S$, then we let a/s denote the equivalence class in $R \times S$ that contains (a, s) . One can check that

$$(17.5) \quad (a/s) + (b/t) = (a t + b s)/(s t)$$

and

$$(17.6) \quad (a/s) (b/t) = (a b)/(s t)$$

are well-defined on $S^{-1} R$, and that $S^{-1} R$ becomes a commutative ring with multiplicative identity element given by e/e . By construction,

$$(17.7) \quad f(a) = a/e$$

defines a homomorphism from R into $S^{-1} R$, which is not necessarily injective.

More precisely, it is easy to see that $f(a) = 0$ for some $a \in R$ if and only if

$$(17.8) \quad a s = 0$$

for some $s \in S$. If R is an integral domain, and if $0 \notin S$, then it follows that f is injective as a mapping from R into $S^{-1} R$. In particular, if R is an integral domain, then $S = R \setminus \{0\}$ is multiplicatively closed, $S^{-1} R$ is the usual field of fractions associated to R , and f is the standard embedding of R into its ring of fractions. Similarly, for any R and S , $S^{-1} R = \{0\}$ if and only if $0 \in S$, since one can apply the previous criterion to $a = e$.

An element x of R is said to be a unit in R if there is a $y \in R$ such that

$$(17.9) \quad x y = e,$$

in which case y is unique and may be denoted x^{-1} . One might also say that a unit x is invertible in R , but one should note that 0 is a unit when $R = \{0\}$. As usual, the collection of units in R is a group with respect to multiplication.

If S is a multiplicatively closed subset of R and $f : R \rightarrow S^{-1} R$ is as before, then $f(s)$ is a unit in $S^{-1} R$ for every $s \in S$. By construction, every element of $S^{-1} R$ can be expressed as $f(a) f(s)^{-1}$ for some $a \in R$ and $s \in S$. If every element of S is a unit in R , then f is an isomorphism from R onto $S^{-1} R$, and of course one did not really need to construct $S^{-1} R$.

As a basic class of examples, let x be an element of R , and let S be the collection of powers x^n of x for each positive integer n , together with e , which corresponds to $n = 0$. If x is a unit in R , then every element of S is a unit in R , and $S^{-1} R$ is isomorphic to R . If R is the ring \mathbf{Z} of integers and $x \neq 0$, then $S^{-1} R$ is the ring of rational numbers with denominator equal to a power of x . If S is any multiplicatively closed set of nonzero integers, then $S^{-1} \mathbf{Z}$ is the ring of rational numbers with denominators in S .

18 Modules of fractions

Let R be a commutative ring with multiplicative identity element e , let S be a multiplicatively closed subset of R , and let $S^{-1}R$ be the corresponding ring of fractions, as in the previous section. If M is a module over R , then there is an analogous construction of $S^{-1}M$, as a module over $S^{-1}R$. More precisely, one can first define a relation \equiv on $M \times S$, by saying that $(x, s) \equiv (x', s')$ for some $x, x' \in M$ and $s, s' \in S$ when

$$(18.1) \quad t(s'x - sx') = 0$$

in M for some $t \in S$. One can check that this is an equivalence relation on $M \times S$, in essentially the same way as before.

Let $S^{-1}M$ be the corresponding collection of equivalence classes in $M \times S$, and let x/s denote the equivalence class that contains (x, s) for every $x \in M$ and $s \in S$. One can define addition of elements of $S^{-1}M$ in the same way as before, as well as multiplication of elements of $S^{-1}M$ by elements of $S^{-1}R$, so that $S^{-1}M$ becomes a module over $S^{-1}R$. Note that $x/e = 0$ in $S^{-1}M$ for some $x \in M$ if and only if

$$(18.2) \quad sx = 0$$

in M for some $s \in S$.

If $M = R$ as a module over itself, for instance, then it is easy to see that $S^{-1}M = S^{-1}R$ as a module over itself. Similarly, if $M = R^n$ for some positive integer n , where addition and scalar multiplication are defined coordinatewise, then $S^{-1}M = (S^{-1}R)^n$.

If M is the direct sum of finitely many modules M_1, \dots, M_n over R , then $S^{-1}M$ is equivalent to the direct sum of $S^{-1}M_1, \dots, S^{-1}M_n$ as modules over $S^{-1}R$. This also works for the direct sum of infinitely many modules over R , but one should be careful about the distinction between the direct sum and the direct product for infinitely many modules. In the direct sum, all but finitely many coordinates of any element are equal to 0, which permits an element of a direct sum of fractions to be expressed with a common denominator.

If M and N are modules over R and $\phi : M \rightarrow N$ is R -linear, then there is a mapping $S^{-1}\phi$ from $S^{-1}M$ into $S^{-1}N$ that sends x/s in $S^{-1}M$ to $\phi(x)/s$ in $S^{-1}N$ for every $x \in M$ and $s \in S$. One can check that this is well-defined and $S^{-1}R$ -linear.

In particular, if M' is a submodule of M , then one can identify $S^{-1}M'$ with a submodule of $S^{-1}M$. One can also verify that $S^{-1}(M/M')$ is isomorphic as a module over $S^{-1}R$ to $S^{-1}M/S^{-1}M'$.

Now let M_1, M_2 , and N be modules over R , and let B be an R -bilinear mapping from $M_1 \times M_2$ into N . Under these conditions, we get a mapping $S^{-1}B$ from $S^{-1}M_1 \times S^{-1}M_2$ into $S^{-1}N$, which satisfies

$$(18.3) \quad (S^{-1}B)(x/s, y/t) = B(x, y)/(st)$$

for every $x \in M_1, y \in M_2$, and $s, t \in S$. Again one can check that this is well-defined and $S^{-1}R$ -bilinear. If $M_1 = M_2 = R^n$ for some positive integer n , and

$N = R$, then every R -linear mapping from $M_1 \times M_2$ into N can be expressed as in (15.3). In this case, $S^{-1}M_1 = S^{-1}M_2 = (S^{-1}R)^n$, $S^{-1}N = S^{-1}R$, and $S^{-1}B$ can be given by an analogous expression, in which the coefficients $b_{j,l} \in R$ are mapped into $S^{-1}R$ in the usual way.

19 Heisenberg groups of fractions

Let R be a commutative ring with multiplicative identity element e , let S be a multiplicative subset of R , and let $S^{-1}R$ be the corresponding ring of fractions, as in Section 17. Also let M and N be modules over R , and let $S^{-1}M$ and $S^{-1}N$ be the corresponding modules of fractions over $S^{-1}R$, as in the previous section. If B is an R -bilinear mapping from $M \times M$ into N , then there is an associated $S^{-1}R$ -bilinear mapping $S^{-1}B$ from $S^{-1}M \times S^{-1}M$ into $S^{-1}N$, as in (18.3). This leads to a group structure on $S^{-1}G = S^{-1}M \times S^{-1}N$ as in Section 16. Let $G = M \times N$ with the group structure associated to B as in Section 16. There is a natural homomorphism from G into $S^{-1}G$, defined as follows. There is a natural mapping from M into $S^{-1}M$, which sends $x \in M$ to x/e in $S^{-1}M$. There is an analogous mapping from N into $S^{-1}N$, which can be combined with the previous one to give a mapping from G into $S^{-1}G$. It is easy to see that this mapping is a homomorphism, because of (18.3). This mapping also intertwines the dilations δ_r on G with their counterparts on $S^{-1}G$, using the canonical homomorphism from R into $S^{-1}R$ described in Section 17.

20 Restriction of scalars

Let R and R_1 be commutative rings, and let f be a homomorphism from R into R_1 . If M_1 is a module over R_1 , then we can also think of M_1 as a module over R , using f . More precisely, multiplication by $r \in R$ on M_1 as a module over R is defined to be the same as multiplication by $f(r)$ on M_1 as a module over R_1 . If R and R_1 have multiplicative identity elements e and e_1 , respectively, then it is customary to require that $f(e) = e_1$, and that multiplication by e_1 correspond to the identity mapping on M_1 as a module over R_1 , so that multiplication by e also corresponds to the identity mapping on M_1 as a module over R . This process for converting M_1 from a module over R_1 into a module over R using f is known as restriction of scalars.

If N_1 is another module over R_1 and ϕ is a homomorphism from M_1 into N_1 as modules over R_1 , then ϕ is also a homomorphism from M_1 into N_1 as modules over R , using restriction of scalars. Similarly, R_1 -bilinear mappings may be considered as R -bilinear mappings using restriction of scalars.

Let R be a commutative ring with multiplicative identity element e again, and let S be a multiplicatively closed subset of R . As in Section 17, this leads to the corresponding ring of fractions $S^{-1}R$, and a natural homomorphism f from R into $S^{-1}R$ associated to the construction of $S^{-1}R$. If M is a module over R , then $S^{-1}M$ may be defined as a module over $S^{-1}R$ as in Section 18. We

can also think of $S^{-1}M$ as a module over R , using f and restriction of scalars. In particular, the natural mapping

$$(20.1) \quad x \mapsto x/e$$

from M into $S^{-1}M$ may be considered as a homomorphism between modules over R in this way.

If N is another module over R , then we can also consider $S^{-1}N$ as a module over R using restriction of scalars. If B is an R -bilinear mapping from $M \times M$ into N , then we get an $S^{-1}R$ -bilinear mapping $S^{-1}B$ from $S^{-1}M \times S^{-1}M$ into $S^{-1}N$, as in Section 18. This leads to a group structure on $S^{-1}G = S^{-1}M \times S^{-1}N$, as in Section 16. The natural mappings from M and N into $S^{-1}M$ and $S^{-1}N$, respectively, combine to give a natural homomorphism from $G = M \times N$ with the group structure associated to B as in Section 16 into $S^{-1}G$, as in the previous section. These natural mappings from M and N into $S^{-1}M$ and $S^{-1}N$ may be considered as homomorphisms between R -modules using restriction of scalars, as in the preceding paragraph, so that the homomorphism from G into $S^{-1}G$ discussed in the previous section can be seen as a homomorphism of the type described in Section 16.

21 A class of examples

Let R be a commutative ring with multiplicative identity element e , and let \mathcal{I} be an ideal in R . It is easy to see that

$$(21.1) \quad S = e + \mathcal{I} = \{e + x : x \in \mathcal{I}\}$$

is a multiplicatively closed subset of R . Note that $0 \notin S$ when $\mathcal{I} \neq R$. Thus $S^{-1}R$ and $f : R \rightarrow S^{-1}R$ can be defined as in Section 17.

Remember that $f(a) = 0$ for some $a \in R$ if and only if $as = 0$ for some $s \in S$. In the present situation, this means that

$$(21.2) \quad a(e + x) = 0$$

for some $x \in \mathcal{I}$. Equivalently,

$$(21.3) \quad a = a(-x),$$

and hence

$$(21.4) \quad a = a(-x)^n$$

for every positive integer n .

If \mathcal{I}' , \mathcal{I}'' are ideals in R , then their product $\mathcal{I}'\mathcal{I}''$ is the ideal in R consisting of all finite sums of products xy of elements x of \mathcal{I}' and y of \mathcal{I}'' . Observe that

$$(21.5) \quad \mathcal{I}'\mathcal{I}'' = \mathcal{I}''\mathcal{I}',$$

because R is commutative, and that

$$(21.6) \quad \mathcal{I}'\mathcal{I}'' \subseteq \mathcal{I}' \cap \mathcal{I}'',$$

because \mathcal{I}' and \mathcal{I}'' are ideals in R . Similarly, the product of n ideals in R is the ideal consisting of all finite sums of products of elements from each of the n ideals, which can also be obtained by taking the products of the ideals one at a time using the previous definition. In particular, the n th power of the ideal \mathcal{I} is defined for each positive integer n as the product of n \mathcal{I} 's. Equivalently, \mathcal{I}^n can be defined recursively by $\mathcal{I}^1 = \mathcal{I}$ and

$$(21.7) \quad \mathcal{I}^{n+1} = \mathcal{I}\mathcal{I}^n$$

for $n \geq 1$.

If $a \in R$ satisfies $f(a) = 0$, as before, then there is an $x \in \mathcal{I}$ such that (21.4) holds for each positive integer n , and hence $a \in \mathcal{I}^n$ for each n . In particular, if $\bigcap_{n=1}^{\infty} \mathcal{I}^n = \{0\}$, then it follows that $f : R \rightarrow S^{-1}R$ is injective.

22 Chains of submodules

Let R be a commutative ring, and let M be a module over R . Suppose that $M_0 = M \supseteq M_1 \supseteq M_2 \supseteq \cdots$ is a decreasing chain of submodules of M such that $\bigcap_{j=1}^{\infty} M_j = \{0\}$. Under these conditions, there is a standard way to define a topology on M , where a subset U of M is an open set if for each $x \in U$ there is a $j \geq 0$ such that

$$(22.1) \quad x + M_j = \{x + y : y \in M_j\} \subseteq U.$$

It is easy to see that this defines a topology on M , and that $x + M_j$ is an open set in M for each $x \in M$ and $j \geq 0$. One can also check that M is Hausdorff with respect to this topology, because of the hypothesis that $\bigcap_{j=1}^{\infty} M_j = \{0\}$.

By construction, this topology is invariant under translations on M . More precisely, M becomes a topological abelian group with respect to addition. This means that $(x, y) \mapsto x + y$ is a continuous mapping from $M \times M$ into M , where $M \times M$ is equipped with the product topology corresponding to the topology on M just described, and that $x \mapsto -x$ is also continuous as a mapping from M onto itself. Similarly, for each $r \in R$, one can check that $x \mapsto rx$ is a continuous mapping on M . Note that the sequence of M_j 's forms a local base for this topology on M at 0.

Let $j(x)$ be the largest nonnegative integer such that $x \in M_{j(x)}$ for each $x \in M$ with $x \neq 0$, and put $j(0) = +\infty$. Thus

$$(22.2) \quad j(-x) = j(x)$$

for every $x \in M$,

$$(22.3) \quad j(x + y) \geq \min(j(x), j(y))$$

for every $x, y \in M$, and

$$(22.4) \quad j(rx) \geq j(x)$$

for every $x \in M$ and $r \in R$. Let $a_0 = 1 \geq a_1 \geq a_2 \geq \cdots$ be a monotone decreasing sequence of positive real numbers that converges to 0, and put

$$(22.5) \quad \rho(x) = a_{j(x)}$$

for each $x \in M$ with $x \neq 0$, and $\rho(0) = 0$. Note that

$$(22.6) \quad \rho(-x) = \rho(x)$$

for every $x \in M$,

$$(22.7) \quad \rho(x + y) \leq \max(\rho(x), \rho(y))$$

for every $x, y \in M$, and

$$(22.8) \quad \rho(rx) \leq \rho(x)$$

for every $x \in M$ and $r \in R$, by the corresponding properties of $j(x)$.

It is easy to see that

$$(22.9) \quad d(x, y) = \rho(x - y)$$

defines a metric on M that determines the same topology on M as described earlier. More precisely, this is an ultrametric on M , in the sense that

$$(22.10) \quad d(x, z) \leq \max(d(x, y), d(y, z))$$

for every $x, y, z \in M$. By construction, this metric is also invariant under translations, in the sense that

$$(22.11) \quad d(x - z, y - z) = d(x, y)$$

for every $x, y, z \in M$. We also have that

$$(22.12) \quad d(rx, ry) \leq d(x, y)$$

for every $x, y \in M$ and $r \in R$, because of (22.8).

23 Completeness

Let R be a commutative ring, let M be a module over R , and let

$$(23.1) \quad M_0 = M \supseteq M_1 \supseteq M_2 \supseteq \cdots$$

be a decreasing chain of submodules of M such that $\bigcap_{j=1}^{\infty} M_j = \{0\}$, as in the previous section. A sequence $\{x_j\}_{j=1}^{\infty}$ of elements of M is said to be a Cauchy sequence in M if for each $n \geq 1$ there is an $L(n) \geq 1$ such that

$$(23.2) \quad x_j - x_l \in M_n$$

for every $j, l \geq L(n)$. This is equivalent to saying that $\{x_j\}_{j=1}^{\infty}$ is a Cauchy sequence with respect to any of the translation-invariant ultrametrics on M discussed in the previous section, or with respect to any translation-invariant metric on M that determines the same topology. As usual, it is easy to see that every convergent sequence in M is a Cauchy sequence in M . Conversely, if every Cauchy sequence in M converges to an element of M , then M is said to be complete.

A pair of Cauchy sequences $\{x_j\}_{j=1}^\infty$ and $\{y_j\}_{j=1}^\infty$ of elements of M is said to be equivalent if $\{x_j - y_j\}_{j=1}^\infty$ converges to 0 in M . One can check that this defines an equivalence relation on the collection of all Cauchy sequences of elements of M . In particular, every subsequence of a Cauchy sequence is also Cauchy sequence, and is equivalent to the original sequence.

Let us say that a sequence $\{x_j\}_{j=1}^\infty$ of elements of M is “strongly Cauchy” if

$$(23.3) \quad x_j - x_l \in M_j$$

for every $l \geq j \geq 1$. Every Cauchy sequence in M has a subsequence which is strongly Cauchy, and hence every Cauchy sequence in M is equivalent to a strongly Cauchy sequence. Observe that two strongly Cauchy sequences $\{x_j\}_{j=1}^\infty$ and $\{y_j\}_{j=1}^\infty$ of elements of M are equivalent as Cauchy sequences if and only if

$$(23.4) \quad x_j - y_j \in M_j$$

for each j . This is the same as saying that x_j and y_j have the same image in M/M_j for each j .

If $j \leq l$, then there is a natural homomorphism $\theta_{j,l}$ from M/M_l onto M/M_j as modules over R , because $M_l \subseteq M_j$. The kernel of this homomorphism is equal to M_l/M_j , and we have that

$$(23.5) \quad \theta_{j,l} \circ \theta_{l,n} = \theta_{j,n}$$

when $n \geq l \geq j$. A sequence $\{\xi_j\}_{j=1}^\infty$ with $\xi_j \in M/M_j$ for each j is said to be a coherent sequence if

$$(23.6) \quad \theta_{j,l}(\xi_l) = \xi_j$$

for every $l \geq j$. Of course, it suffices to check this with $l = j + 1$ for each j , because of (23.5).

Note that every sequence $\{\xi_j\}_{j=1}^\infty$ with $\xi_j \in M/M_j$ for each $j \geq 1$ can be expressed as $\{q_j(x_j)\}_{j=1}^\infty$ for some sequence $\{x_j\}_{j=1}^\infty$ of elements of M , where q_j is the natural quotient homomorphism from M onto M/M_j as modules over R for each j . It is easy to see that a sequence $\{x_j\}_{j=1}^\infty$ of elements of M is strongly Cauchy if and only if $\{q_j(x_j)\}_{j=1}^\infty$ is a coherent sequence. Similarly, two strongly Cauchy sequences $\{x_j\}_{j=1}^\infty$ and $\{y_j\}_{j=1}^\infty$ of elements of M are equivalent as Cauchy sequences in M if and only if

$$(23.7) \quad q_j(x_j) = q_j(y_j)$$

for every j , as in (23.4).

If $\{x_j\}_{j=1}^\infty$ is any Cauchy sequence of elements of M , then $\{q_n(x_j)\}_{j=1}^\infty$ is eventually constant in M/M_n for each n . If ξ_n is the limiting value of $q_n(x_j)$ as $j \rightarrow \infty$ for each n , then one can check that $\{\xi_n\}_{n=1}^\infty$ is a coherent sequence. If $\{y_j\}_{j=1}^\infty$ is another Cauchy sequence of elements of M , then $\{y_j\}_{j=1}^\infty$ is equivalent to $\{x_j\}_{j=1}^\infty$ if and only if the corresponding coherent sequences are the same. Thus there is a natural one-to-one correspondence between equivalence classes of Cauchy sequences and coherent sequences. This correspondence is a bit simpler

when we restrict our attention to strongly Cauchy sequences in M , as in the preceding paragraph.

It is customary to define the completion \widehat{M} of M to be the set of equivalence classes of Cauchy sequences in M . With this definition, there is a natural embedding of M into \widehat{M} , which sends each element x of M to the equivalence class of Cauchy sequences containing the constant sequence $\{x_j\}_{j=1}^\infty$ with $x_j = x$ for each j . Alternatively, the completion of M can be identified with the set of coherent sequences, as in the previous paragraphs. The standard embedding of M into the completion can then be given by sending each $x \in M$ to

$$(23.8) \quad q(x) = \{q_j(x)\}_{j=1}^\infty,$$

which is easily seen to be a coherent sequence.

Let X be the Cartesian product $\prod_{j=1}^\infty (M/M_j)$, consisting of the sequences $\xi = \{\xi_j\}_{j=1}^\infty$ with $\xi_j \in M/M_j$ for each j . This can also be considered as a module over R , where addition and multiplication by elements of R are defined termwise. The mapping q from M into X given by (23.8) is a homomorphism of M into X as modules over R , which is injective because of the hypothesis that $\bigcap_{j=1}^\infty M_j = \{0\}$. The set of coherent sequences forms a submodule of X that contains $q(M)$, so that the completion \widehat{M} of M may be considered as a module over R . This is compatible with the usual extension of addition and scalar multiplication from M to \widehat{M} in terms of Cauchy sequences.

We can also consider X as a topological space, using the product topology associated to the discrete topology on M/M_j for each j . It is easy to see that X is a topological abelian group with respect to addition, and that multiplication by r defines a continuous mapping on X for each $r \in R$. One can check that q is a homeomorphism from M onto $q(M)$ with respect to the topology induced on $q(M)$ by the one on X , and that the set of coherent sequences is a closed subset of X . More precisely, the set of coherent sequences is the same as the closure $\overline{q(M)}$ of $q(M)$ in X , which we identify with the completion \widehat{M} of M .

If n is a nonnegative integer, then let X_n be the set of $\xi \in X$ such that

$$(23.9) \quad \xi_j = 0$$

for each $j \leq n$. Thus $X_0 = X$, $X_{n+1} \subseteq X_n$ for each n , X_n is a submodule of X for each n , and $\bigcap_{n=1}^\infty X_n = \{0\}$. This brings us back to the same situation as for M in the previous section, and it is easy to see that the topology on X determined by the X_n 's as before is the same as the product topology associated to the discrete topology on M/M_j for each j . One can also check that X is automatically complete, basically because Cauchy sequences of elements of X converges termwise.

Let \widehat{M}_n be the closure $\overline{q(M_n)}$ of $q(M_n)$ in X for each $n \geq 0$, which is the same as the intersection of $\widehat{M} = \overline{q(M)}$ with X_n for each n . As before, $\widehat{M}_0 = \widehat{M} = \overline{q(M)}$, \widehat{M}_n is a submodule of \widehat{M} for each n , $\widehat{M}_{n+1} \subseteq \widehat{M}_n$ for each n , and $\bigcap_{n=1}^\infty \widehat{M}_n = \{0\}$. This brings us back again to the same situation as for M in the previous section, and it is easy to see that the topology on \widehat{M}

determined by the \widehat{M}_n 's is the same as the one induced by the topology on X already defined. Note that \widehat{M} is complete, as it should be.

We can also consider translation-invariant ultrametrics on X that determine the same topology on X , as in the previous section. If we use the same sequence of a_j 's as for M , then q is an isometric embedding of M into X , and $\widehat{M} = \overline{q(M)}$ corresponds naturally to the completion of M as a metric space too.

If M/M_j has only finitely many elements for each j , then X is compact, and it follows that $\widehat{M} = \overline{q(M)}$ is compact as well. In this case, it is easy to see that M was already totally bounded with respect to the translation-invariant ultrametrics discussed in the previous section.

24 Chains of ideals

Let R be a commutative ring, and suppose that $\mathcal{I}_0 = R \supseteq \mathcal{I}_1 \supseteq \mathcal{I}_2 \supseteq \cdots$ is a decreasing chain of ideals in R such that $\bigcap_{j=1}^{\infty} \mathcal{I}_j = \{0\}$. In particular, we can think of R as a module over itself, and we can think of the \mathcal{I}_j 's as submodules of R , so that everything in the previous two sections can be used here.

Of course, we have more structure now, and it is easy to see for instance that R is a topological ring with respect to the topology defined earlier. This means that R is a topological abelian group with respect to addition, as discussed before, and that multiplication on R is continuous as a mapping from $R \times R$ into R , using the product topology on $R \times R$ associated to the topology already defined on R . If $j(x)$ is defined for $x \in R$ as in Section 22, then we have that

$$(24.1) \quad j(xy) \geq \max(j(x), j(y))$$

for every $x, y \in R$, in place of (22.4). This corresponds to the fact that

$$(24.2) \quad xy \in \mathcal{I}_{j(x)} \cap \mathcal{I}_{j(y)}$$

for every $x, y \in R \setminus \{0\}$. Similarly, if $\rho(x)$ is defined for $x \in R$ as in Section 22, using a monotone decreasing sequence $\{a_j\}_{j=1}^{\infty}$ of positive real numbers as before, then we have that

$$(24.3) \quad \rho(xy) \leq \min(\rho(x), \rho(y))$$

for every $x, y \in R$, instead of (22.8).

The completion \widehat{R} of R can be described as in the preceding section. The main difference is that multiplication on R can be extended to \widehat{R} , so that \widehat{R} is a commutative ring as well. More precisely,

$$(24.4) \quad X = \prod_{j=1}^{\infty} (R/\mathcal{I}_j)$$

is a commutative ring in this case, with respect to coordinatewise addition and multiplication, and using the standard ring structure on the quotient R/\mathcal{I}_j for

each j . If we identify \widehat{R} with the closure $\overline{q(R)}$ of the image $q(R)$ of the usual embedding q of R into X , which is the same as the set of coherent sequences, then \widehat{R} becomes a subring of X . This is compatible with the usual extension of multiplication from R to \widehat{R} in terms of Cauchy sequences. Note that X is actually a topological ring with respect to the product topology associated to the discrete topology on R/\mathcal{I}_j for each j , and hence that $\widehat{R} = \overline{q(R)}$ is also a topological ring with respect to the induced topology. These topologies on X and \widehat{R} can be described in terms of chains of ideals, as in the previous section.

Suppose that R is a commutative ring with multiplicative identity element e , and that \mathcal{I} is a proper ideal in R . Put $\mathcal{I}_0 = R$, and let \mathcal{I}_n be the n th power \mathcal{I}^n of \mathcal{I} for each positive integer n , as in Section 21. Thus

$$(24.5) \quad \mathcal{I}_{n+1} \subseteq \mathcal{I}_n$$

for each n by construction, but $\bigcap_{n=1}^{\infty} \mathcal{I}_n = \{0\}$ is an additional hypothesis. Let $x \in \mathcal{I}$ be given, and put

$$(24.6) \quad s_n = \sum_{j=0}^n x^j$$

for each n , where $x^j = e$ when $j = 0$. It is easy to see that $\{s_n\}_{n=1}^{\infty}$ is a strongly Cauchy sequence in R with respect to this chain of ideals, since $x^n \in \mathcal{I}_n$ for each n . We also have that

$$(24.7) \quad (e - x) s_n = e - x^{n+1}$$

for each n , by elementary algebra. It follows that $e - x$ is invertible in the completion \widehat{R} of R , with inverse equal to the limit of the s_n 's in \widehat{R} . Similarly, if x is an element of the closure $\widehat{\mathcal{I}}$ in \widehat{R} , which can be identified with the closure $\overline{q(\mathcal{I})}$ of $q(\mathcal{I})$ in X , then $e - x$ is also invertible in \widehat{R} .

25 Ideals and submodules

Let R be a commutative ring, and let M be a module over R . If \mathcal{I} is an ideal in R , then the product $\mathcal{I}M$ is defined to be the collection of all finite sums of products of the form ax , where $a \in \mathcal{I}$ and $x \in M$. It is easy to see that this is a submodule of M .

Suppose that $\mathcal{I}_0 = R \supseteq \mathcal{I}_1 \supseteq \mathcal{I}_2 \supseteq \cdots$ is a decreasing chain of ideals in R such that $\bigcap_{j=1}^{\infty} \mathcal{I}_j = \{0\}$, and that $M_0 = M \supseteq M_1 \supseteq M_2 \supseteq \cdots$ is a decreasing chain of submodules of M such that $\bigcap_{j=1}^{\infty} M_j = \{0\}$. As a simple compatibility condition between these two chains, let us ask that

$$(25.1) \quad \mathcal{I}_j M \subseteq M_j$$

for each j . This implies that multiplication of elements of M by elements of R defines a continuous mapping from $R \times M$ into M with respect to the topologies on M and R discussed in Sections 22 and 24, and using the corresponding

product topology on $R \times M$. If $j_M(x)$ and $j_R(r)$ are defined for $x \in M$ and $r \in R$ as before, then we get that

$$(25.2) \quad j_M(rx) \geq \max(j_R(r), j_M(x))$$

for every $r \in R$ and $x \in M$, in place of (22.4). Similarly, if $\rho_M(x)$ and $\rho_R(r)$ are defined for $x \in M$ and $r \in R$ as before, using the same monotone decreasing sequence $\{a_j\}_{j=1}^\infty$ of positive real numbers, then we get that

$$(25.3) \quad \rho_M(rx) \leq \min(\rho_R(r), \rho_M(x))$$

for every $r \in R$ and $x \in M$, instead of (22.8).

Note that (25.1) is equivalent to saying that the product of any element of \mathcal{I}_j with any element of M/M_j as a module over R is equal to 0. This implies that M/M_j may also be considered as a module over R/\mathcal{I}_j for each j . If

$$(25.4) \quad X_M = \prod_{j=1}^\infty (M/M_j) \quad \text{and} \quad X_R = \prod_{j=1}^\infty (R/\mathcal{I}_j)$$

are as in the previous sections, then it follows that X_M is a module over X_R with respect to coordinatewise multiplication. Moreover, multiplication of elements of X_M by elements of X_R defines a continuous mapping from $X_R \times X_M$ into X_M with respect to the product topologies on X_M and X_R associated to the discrete topologies on M/M_j and R/\mathcal{I}_j for each j , and using the corresponding product topology on $X_R \times X_M$.

In this situation, the completion \widehat{M} of M may be considered as a module over the completion \widehat{R} of R . The completions \widehat{M} and \widehat{R} of M and R may be identified with the subsets of X_M and X_R consisting of coherent sequences, respectively, and one can check that the product of a coherent sequence in X_M by a coherent sequence in X_R is a coherent sequence in X_M . As usual, this is compatible with extending scalar multiplication from M and R to \widehat{M} and \widehat{R} using Cauchy sequences under these conditions. As before, multiplication of elements of \widehat{M} by elements of \widehat{R} defines a continuous mapping from $\widehat{R} \times \widehat{M}$ into \widehat{M} with respect to the appropriate topologies.

26 Homomorphisms

Let R be a commutative ring, let M and N be modules over R , and let f be a homomorphism from M into N . Suppose also that $M_0 = M \supseteq M_1 \supseteq M_2 \supseteq \cdots$ and $N_0 = N \supseteq N_1 \supseteq N_2 \supseteq \cdots$ are decreasing chains of submodules of M and N , respectively, such that $\bigcap_{j=1}^\infty M_j = \{0\}$ and $\bigcap_{j=1}^\infty N_j = \{0\}$. As in Section 22, these chains of submodules determine topologies on M and N . If $f : M \rightarrow N$ is continuous at 0, then for each $j \geq 1$ there is an $l(j) \geq 1$ such that

$$(26.1) \quad f(M_{l(j)}) \subseteq N_j,$$

because N_j is a neighborhood of 0 in N . Conversely, this condition implies that f is continuous at every point in M , because f is a homomorphism and

the topologies on M and N are invariant under translations. More precisely, if $x, y \in M$ and $x - y \in M_{l(j)}$, then (26.1) implies that

$$(26.2) \quad f(x) - f(y) = f(x - y) \in N_j,$$

because $f : M \rightarrow N$ is a homomorphism. This is basically a uniform continuity condition for f . If M and N are equipped with translation-invariant metrics that determine the same topologies, for instance, then f is uniformly continuous with respect to these metrics on M and N . In particular, this implies that f maps Cauchy sequence in M to Cauchy sequences in N , and that f can be extended to a uniformly continuous mapping \widehat{f} from the completion \widehat{M} of M into the completion \widehat{N} of N . It is easy to see that \widehat{f} is also a homomorphism from \widehat{M} into \widehat{N} as modules over R .

This is all a bit simpler if (26.1) holds with $l(j) = j$, so that

$$(26.3) \quad f(M_j) \subseteq N_j$$

for each j . In this case, f induces a homomorphism f_j from M/M_j into N/N_j as modules over R for each j . As usual,

$$(26.4) \quad X_M = \prod_{j=1}^{\infty} (M/M_j) \quad \text{and} \quad X_N = \prod_{j=1}^{\infty} (N/N_j)$$

may be considered as modules over R , where addition and multiplication by elements of R are defined coordinatewise. Thus f determines a homomorphism f_X from X_M into X_N as modules over R , using the induced homomorphisms f_j on each factor. Note that f_X is also continuous with respect to the product topologies on X_M and X_N associated to the discrete topologies on M/M_j and N/N_j for each j . If we identify the completions \widehat{M} and \widehat{N} with the subsets of X_M and X_N consisting of coherent sequences, respectively, then the extension \widehat{f} of f to a mapping from \widehat{M} into \widehat{N} corresponds to the restriction of f_X to the set of coherent sequences in X_M . In particular, it is easy to see that f_X sends coherent sequences in X_M to coherent sequences in X_N .

Suppose in addition that $\mathcal{I}_0 = R \supseteq \mathcal{I}_1 \supseteq \mathcal{I}_2 \supseteq \cdots$ is a decreasing chain of ideals in R such that $\bigcap_{j=1}^{\infty} \mathcal{I}_j = \{0\}$, and that $\mathcal{I}_j M \subseteq M_j$ and $\mathcal{I}_j N \subseteq N_j$ for each j , as in the previous section. This implies that M/M_j and N/N_j are also modules over R/\mathcal{I}_j for each j , and one can check that f_j is a homomorphism from M/M_j into N/N_j as modules over R/\mathcal{I}_j for each j too. As in the previous section, X_M and X_N may be considered as modules over $X_R = \prod_{j=1}^{\infty} (R/\mathcal{I}_j)$, and now we get that f_X is a homomorphism from X_M into X_N as modules over X_R as well. Under these conditions, we get that the extension \widehat{f} of f to a mapping from the completion \widehat{M} of M to the completion \widehat{N} of N is a homomorphism from \widehat{M} into \widehat{N} as modules over the completion \widehat{R} of R . The same conclusion could also be obtained using the weaker continuity condition (26.1) and Cauchy sequences, and one could weaken the compatibility condition between the ideals \mathcal{I}_j and the submodules M_j and N_j analogously, but some of the other steps would not work in the same way.

As another variant, M and N may be commutative rings themselves, with decreasing chains of ideals M_j and N_j as before. If f is now a continuous homomorphism from M into N , then f extends to a continuous homomorphism \hat{f} from the completion \hat{M} of M to the completion \hat{N} of N as commutative rings. This uses the fact that products of Cauchy sequences in M are still Cauchy sequences in M , which are mapped by f to the corresponding products of Cauchy sequences in N . The description of f is more elegant when f satisfies the stronger continuity condition (26.3), so that $f_j : M/M_j \rightarrow N/N_j$ is a ring homomorphism for each j . This implies that f_X is ring homomorphism from X_M into X_N , and \hat{f} may be identified with the restriction of f_X to the set of coherent sequences in X_M .

27 Bilinear mappings

Let R be a commutative ring, let M and N be modules over R , and let B be an R -bilinear mapping from $M \times M$ into N . One could also consider bilinear mappings defined on products of different modules over R , but this is will not be needed here. Suppose that

$$(27.1) \quad M_0 = M \supseteq M_1 \supseteq M_2 \supseteq \cdots \quad \text{and} \quad N_0 = N \supseteq N_1 \supseteq N_2 \supseteq \cdots$$

are decreasing chains of submodules of M and N , respectively, such that

$$(27.2) \quad \bigcap_{j=1}^{\infty} M_j = \{0\} \quad \text{and} \quad \bigcap_{j=1}^{\infty} N_j = \{0\},$$

as usual. As a simple compatibility condition, let us ask that

$$(27.3) \quad B(x, y) \in N_j$$

for every $x \in M$ and $y \in M_j$, and for every $x \in M_j$ and $y \in M$, for each $j \geq 1$. This is the same as saying that $B(x, y)$ satisfies the continuity condition (26.3) as a function of x for each $y \in M$, and also as a function of y for each $x \in M$.

It is easy to see that B induces an R -bilinear mapping

$$(27.4) \quad B_j : (M/M_j) \times (M/M_j) \rightarrow N/N_j$$

for each j , and hence an R -bilinear mapping

$$(27.5) \quad B_X : X_M \times X_M \rightarrow X_N,$$

where $X_M = \prod_{j=1}^{\infty} (M/M_j)$ and $X_N = \prod_{j=1}^{\infty} (N/N_j)$, as before. One can also check that B_X sends pairs of coherent sequences in X_M to coherent sequences in X_N . If we identify the completions \hat{M} and \hat{N} with the subsets of X_M and X_N consisting of coherent sequences, respectively, then the restriction of B_X to coherent sequences in X_M defines a natural extension of B to an R -bilinear

mapping \widehat{B} from $\widehat{M} \times \widehat{M}$ into \widehat{N} . This is equivalent to extending B to an R -bilinear mapping \widehat{B} from $\widehat{M} \times \widehat{M}$ into \widehat{N} using Cauchy sequences, which also works under less stringent continuity conditions. More precisely, one can use continuity conditions like these on B to show that B sends pairs of Cauchy sequences in M to Cauchy sequences in N , and that equivalent pairs of Cauchy sequences in M are sent to equivalent Cauchy sequences in N , and so on.

Suppose now that $\mathcal{I}_0 = R \supseteq \mathcal{I}_1 \supseteq \mathcal{I}_2 \supseteq \cdots$ is a decreasing chain of ideals in R such that $\bigcap_{j=1}^{\infty} \mathcal{I}_j = \{0\}$, and that $\mathcal{I}_j M \subseteq M_j$ and $\mathcal{I}_j N \subseteq N_j$ for each j , as in the previous sections. Thus M/M_j and N/N_j are also modules over R/\mathcal{I}_j for each j , and one can check that B_j is R/\mathcal{I}_j -bilinear for each j too. As before, X_M and X_N may be considered as modules over $X_R = \prod_{j=1}^{\infty} (R/\mathcal{I}_j)$, and it is easy to see that B_X is X_R -bilinear. As in Section 25, the completions \widehat{M} and \widehat{N} of M and N may be considered as modules over the completion \widehat{R} of R in this case, and one can check that \widehat{B} is \widehat{R} -bilinear. This can be verified more directly in terms of Cauchy sequences as well, using the fact that the product of a Cauchy sequence in M or N with a Cauchy sequence in R is still a Cauchy sequence in M or N , respectively, and so on.

Let \diamond be the group structure on $G = M \times N$ associated to B as in Section 16. The hypothesis (27.3) implies that $H_j = M_j \times N_j$ is a normal subgroup of G for each j , as before. Similarly, there is a group structure \diamond_j on

$$(27.6) \quad G_j = (M/M_j) \times (N/N_j)$$

associated to B_j for each j . Let

$$(27.7) \quad \Phi_j : G \rightarrow G_j$$

be the obvious quotient mapping, obtained by combining the canonical quotient mappings from M onto M/M_j and N onto N/N_j . As in Section 16, Φ_j is a homomorphism from G onto G_j for each j , with kernel equal to H_j .

28 Stronger conditions

Let us continue with the same hypotheses and notations as in the previous section. As a stronger version of (27.3), let us ask that

$$(28.1) \quad B(x, y) \in N_{j+l}$$

for every $x \in M_j$, $y \in M_l$, and $j, l \geq 0$. This implies that

$$(28.2) \quad E_j = M_j \times N_{2j}$$

is a subgroup of $G = M \times N$ with respect to \diamond for each $j \geq 1$. Note that

$$(28.3) \quad H_{2j} \subseteq E_j \subseteq H_j$$

for each j , but that E_j is not necessarily a normal subgroup of G .

Suppose that $\mathcal{I}_0 = R \supseteq \mathcal{I}_1 \supseteq \mathcal{I}_2 \supseteq \cdots$ is a decreasing chain of ideals in R such that $\bigcap_{j=1}^{\infty} \mathcal{I}_j = \{0\}$ and

$$(28.4) \quad \mathcal{I}_j \mathcal{I}_l \subseteq \mathcal{I}_{j+l}$$

for every $j, l \geq 0$. Note that (28.4) holds automatically when \mathcal{I}_j is equal to the j th power \mathcal{I}^j of some ideal \mathcal{I} in R for each $j \geq 1$. As a stronger version of the usual compatibility conditions

$$(28.5) \quad \mathcal{I}_j M \subseteq M \quad \text{and} \quad \mathcal{I}_j N \subseteq N,$$

let us ask that

$$(28.6) \quad \mathcal{I}_j M_l \subseteq M_{j+l} \quad \text{and} \quad \mathcal{I}_j N_l \subseteq N_{j+l}$$

for each $j, l \geq 0$. This would follow from (28.4) if $M_l = \mathcal{I}_l M$ and $N_l = \mathcal{I}_l N$ for each l , as would (28.1). Under these conditions, we get that

$$(28.7) \quad \delta_r(E_l) \subseteq E_{j+l}$$

for every $r \in \mathcal{I}_j$ and $j, l \geq 0$, where δ_r is as in (16.3).

Let us look at how these various conditions extend to the corresponding completions. If \widehat{M} is the completion of M , and \widehat{M}_j is the closure of M_j in \widehat{M} , then \widehat{M}_j is a submodule of \widehat{M} with $\widehat{M}_{j+1} \subseteq \widehat{M}_j$ for each j , and $\bigcap_{j=1}^{\infty} \widehat{M}_j = \{0\}$. As usual, it is easier to deal with completions in terms of coherent sequences, although one can also work directly with Cauchy sequences. Similar remarks apply to N and the N_j 's, and the extension \widehat{B} of B to an R -bilinear mapping from $\widehat{M} \times \widehat{M}$ into \widehat{N} was discussed in the previous section. In the context of the previous section, \widehat{B} would satisfy the analogue of (27.3) for \widehat{M}_j and \widehat{N}_j , and here \widehat{B} satisfies the analogue of (28.1) for \widehat{M}_j and \widehat{N}_j . In the same way, if $\widehat{\mathcal{I}}_j$ is the closure of \mathcal{I}_j in the completion \widehat{R} of R , then $\widehat{\mathcal{I}}_j$ is an ideal in \widehat{R} such that $\widehat{\mathcal{I}}_{j+1} \subseteq \widehat{\mathcal{I}}_j$ for each j , and $\bigcap_{j=1}^{\infty} \widehat{\mathcal{I}}_j = \{0\}$. As before, \widehat{M} and \widehat{N} may be considered as modules over \widehat{R} , and \widehat{B} is \widehat{R} -bilinear. It is easy to see that (28.4) and (28.6) imply their counterparts for $\widehat{\mathcal{I}}_j$, \widehat{M}_j , and \widehat{N}_j as well. If $\widehat{G} = \widehat{M} \times \widehat{N}$, then we can extend the group structure \diamond on G to the group structure $\widehat{\diamond}$ on \widehat{G} associated to \widehat{B} as in Section 16, and $\widehat{H}_j = \widehat{M}_j \times \widehat{N}_j$ is a normal subgroup of \widehat{G} for each j . Under the conditions of this section, $\widehat{E}_j = \widehat{M}_j \times \widehat{N}_{2j}$ is subgroup of \widehat{G} that satisfies the counterparts of (28.3) and (28.7) with \widehat{H}_j and $\widehat{\mathcal{I}}_j$ instead of H_j and \mathcal{I}_j .

As usual, one can get basic examples by taking R to be the ring \mathbf{Z} of integers, and \mathcal{I} to be the ideal $a\mathbf{Z}$ of integer multiples of some integer $a \geq 2$. If \mathcal{I}_j is the j th power \mathcal{I}^j of \mathcal{I} when $j \geq 1$, then \mathcal{I}_j is the same as the ideal $a^j\mathbf{Z}$ of integer multiples of a^j . In this case, the completion \widehat{R} of R is the same as the a -adic completion of \mathbf{Z} .

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